Quantum Calorons

In the CP^{n-1} model

(1) Background:

- Theta terms
- Finite temp QFT

(2) CP^{N-1} Model

- Topological term
- Zero temperature results
- Large N expansion

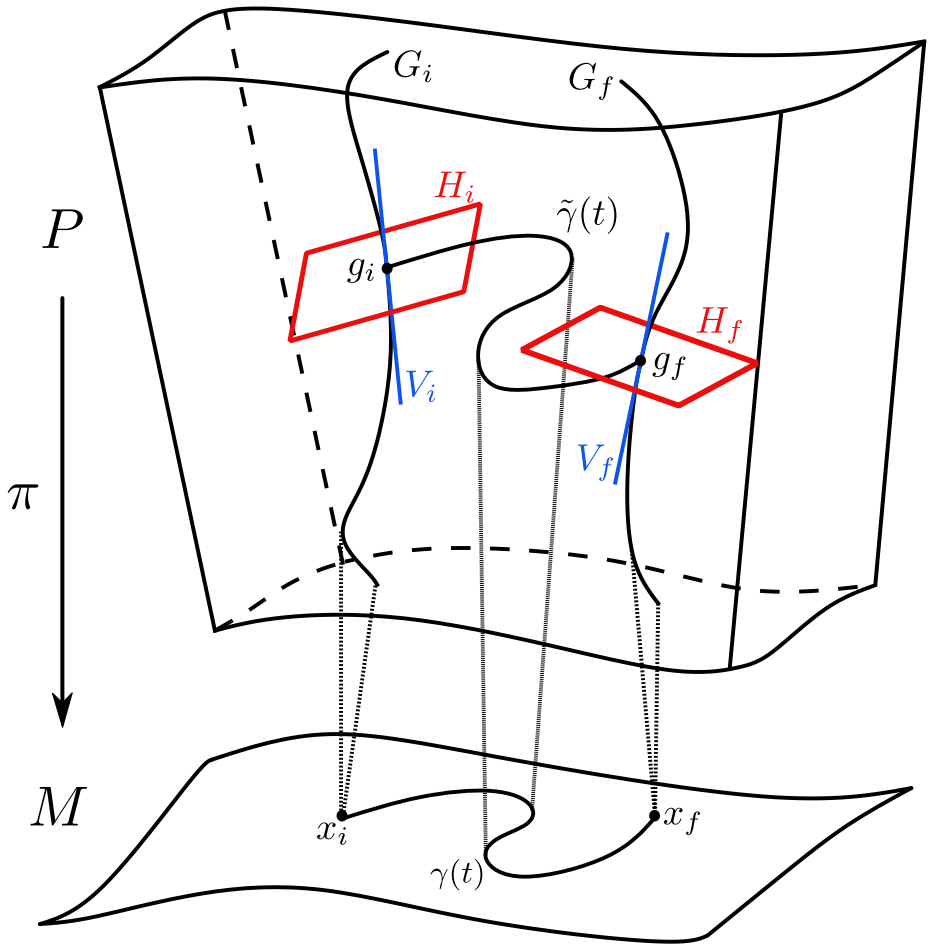
(3) T>0 results (Affleck):

- Nonzero mass
- Calorons
- Theta dependence

(4) Physical Relevance

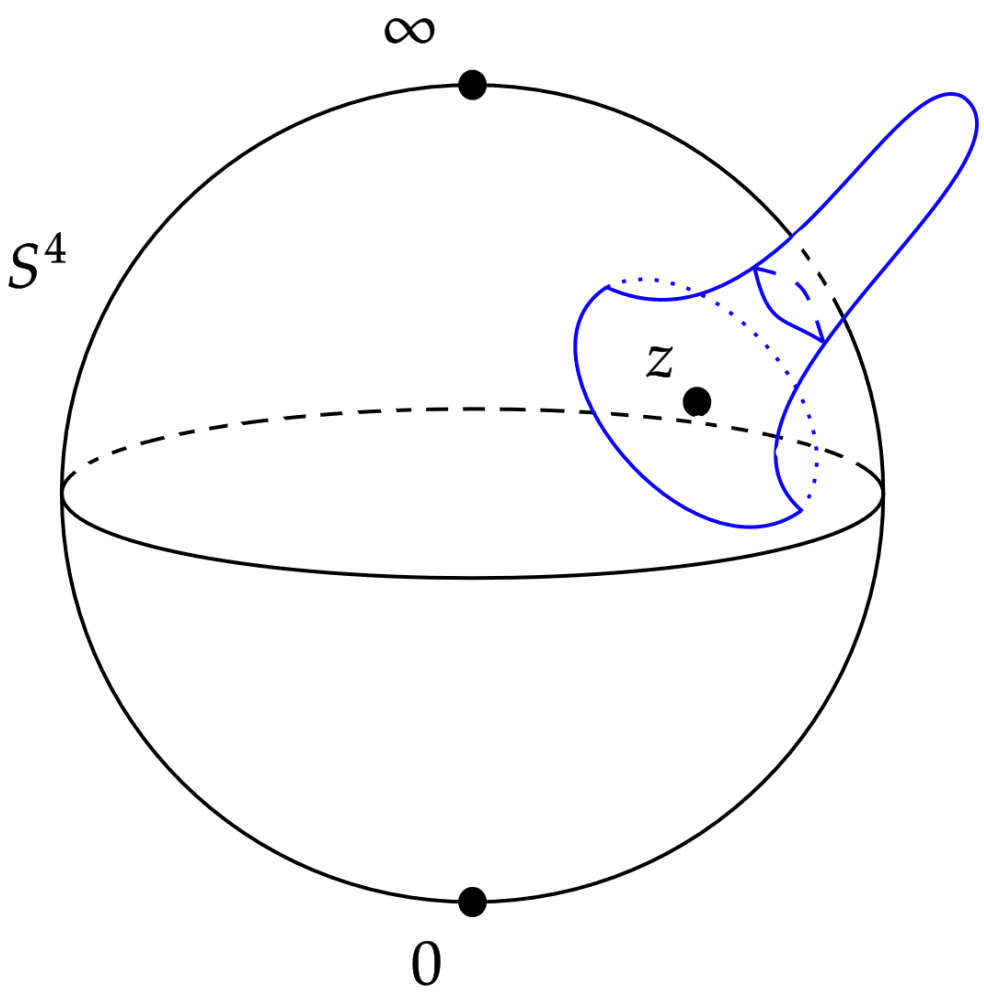
- Effective QED
- Comments on QCD

Theta terms



- Classical gauge theory: Connection lives in fiber bundle over spacetime
- For compact domains, fiber bundles can have distinct topologies (up to homotopy)
- Spacetime is "compactified" by pure gauge condition, or equivalently, get map from "sphere at infinity into gauge group
- Bundles are classified by a Chern number

Theta terms



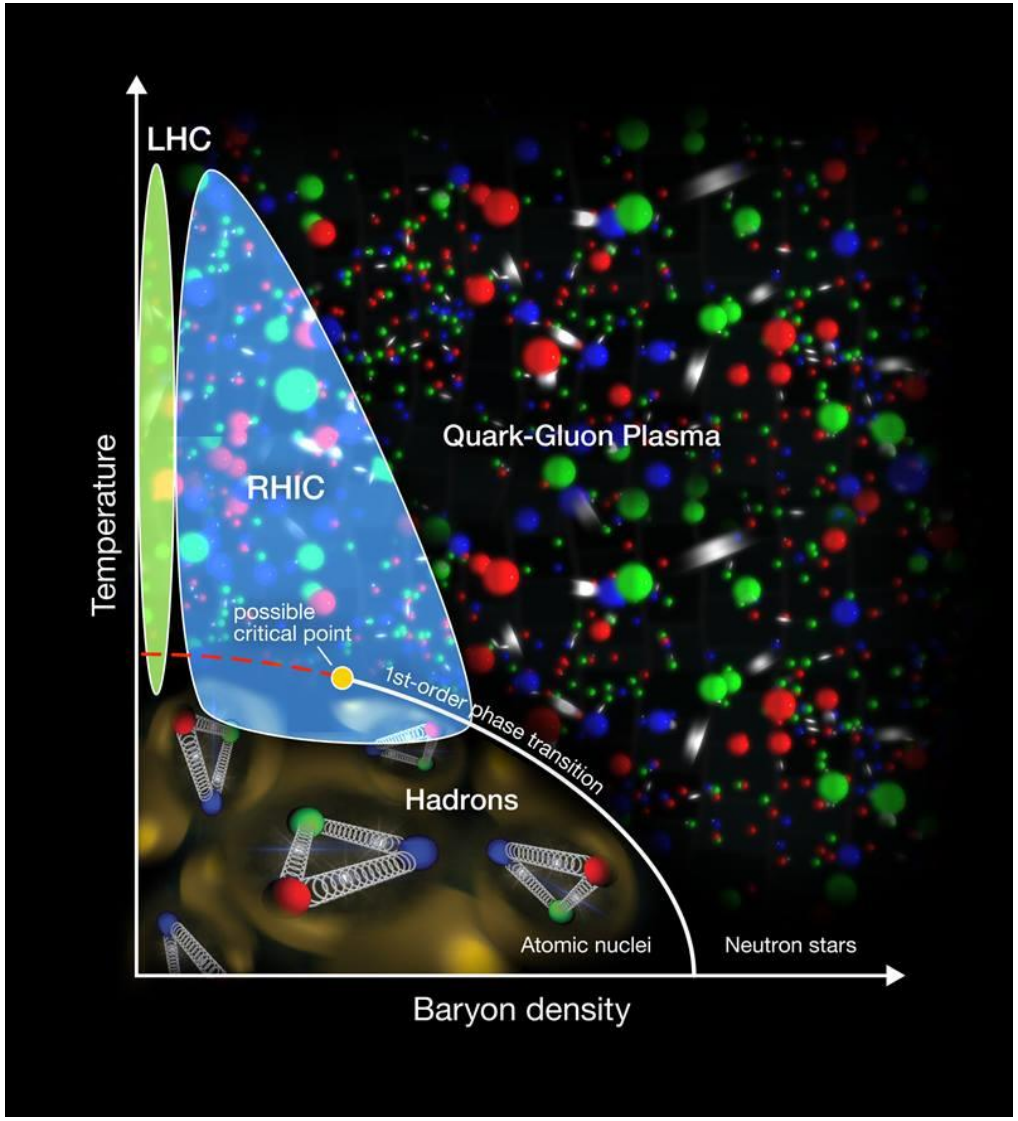
- Get topological charge Q
- Instantons – classical configurations with nonzero Q - "tunneling" between topologies

$$Q = \frac{1}{2\pi}\int_{\partial\mathbb{R}^2} A_\mu$$
$$= \frac{1}{2\pi}\int dx A_1(\tau \to \infty) - \frac{1}{2\pi}\int dx A_1(\tau \to -\infty)$$
$$=: n_\infty - n_{-\infty}$$

- Can construct "block states" which are invariant under transitions between topologically distinct states – VACUA!

Finite Temperature QFT

- Treats background of theory as **finite temperature heat bath**
- Particularly relevant for cosmological calculations (quark gluon plasma, heavy ions, condensed matter)
- Leads to fundamentally different **topological** problem

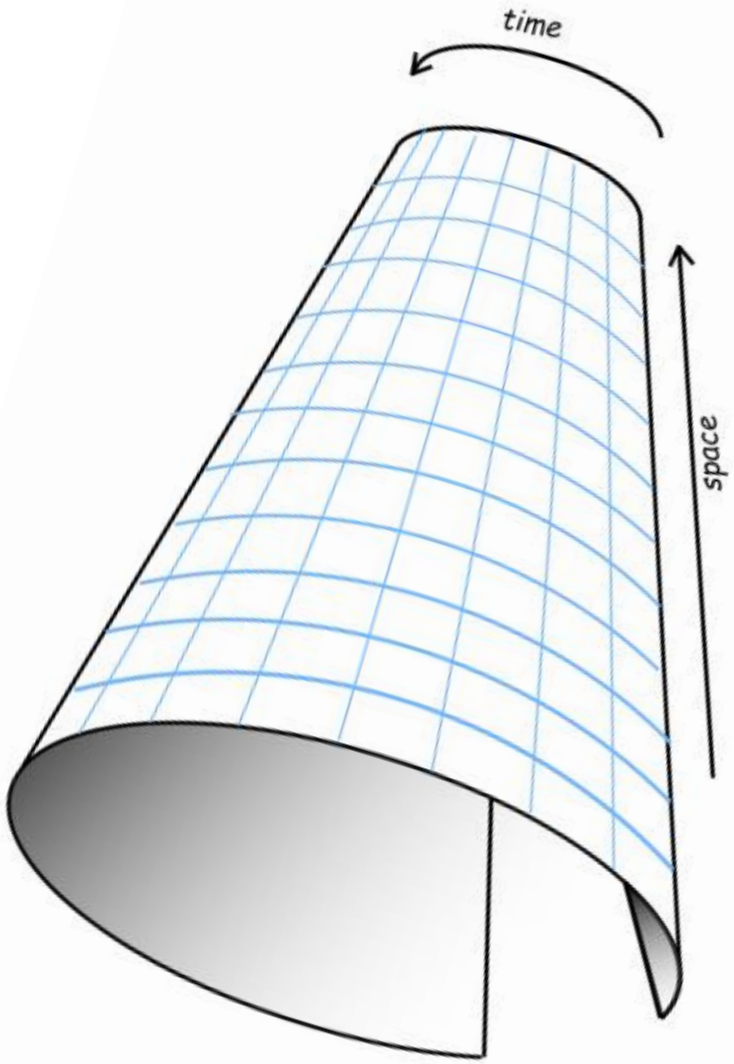


Finite Temperature QFT

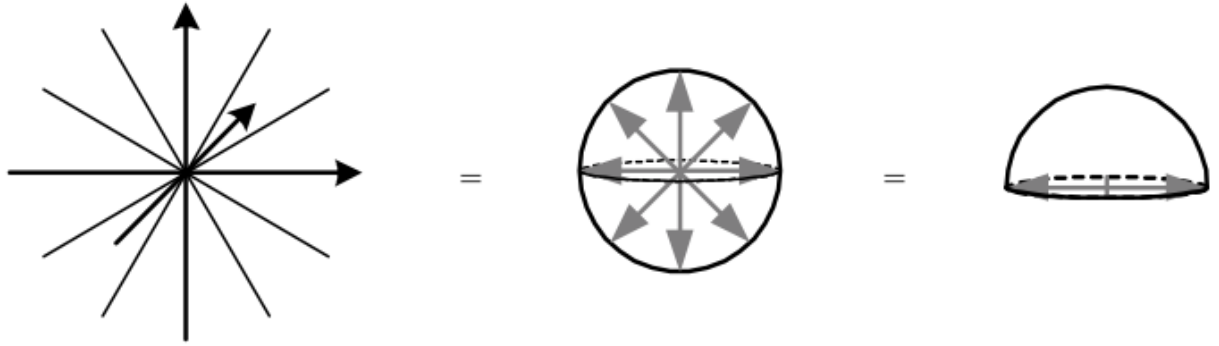
- Partition function is computed via a trace
- Integral in "imaginary time" becomes temperature integral to beta – lives on cylinder!
- KMS condition requires that operators are cyclic

$$\langle A(t)B(0)\rangle = \langle B(0)A(t+i\beta)\rangle$$

- **Fourier transform in time gives SUM**, not integral



The CP^{N-1} Model



- The CP^{N-1} model is a QFT "**valued in CP^{N-1}**"; little n is our field
- It has SU(N) symmetry (n is constrained to |n|^2=1)
- It also has a **U(1) gauge symmetry,** making it valued in the "space of lines"

$$\mathcal{L} = \frac{N}{g^2}|D_\mu n|^2$$

$$A_\mu = \frac{i}{2}(n^*\partial_\mu n - n\partial_\mu n^*)$$

Topological Term of CP^{n-1} Model

- Take background to be \R^2
- Finite action requires connection to become pure gauge
- Get **Bogomolny decomposition**

$$\int d^4x \, |D_\mu n|^2 = \int d^4x \, [\frac{1}{2}|D_\mp n|^2 \pm \frac{i}{2}q]$$

- Get precise form for q given by

$$Q = \frac{\epsilon_{\mu\nu}}{2\pi} \int d^2x \, \partial_\mu A_\nu$$

- Gives us theta term!

T=0 Instantons

- Instantons satisfy **"holomorphicity"** constraint for gauge covariant derivative.
- Fixing gauge, we get a standard (anti-)holomorphicity constraint
- Instantons of T=0 are normalizations of holomorphic solutions; some localized solution on top of pure gauge background.

$$D_{\mp} n = 0$$

Large N expansion

$$S[n,\lambda] = \int d^2x \frac{N}{g^2}|D_\mu n|^2 \mapsto \int d^2x \frac{N}{g^2}\left[|D_\mu n|^2 + i(|n|^2-1)\lambda(x)\right]$$

$$= N\,\mathrm{tr}\ln(D^2 + M^2 + i\lambda) - i\frac{N}{g^2}\int d^2x\,\lambda(x)$$

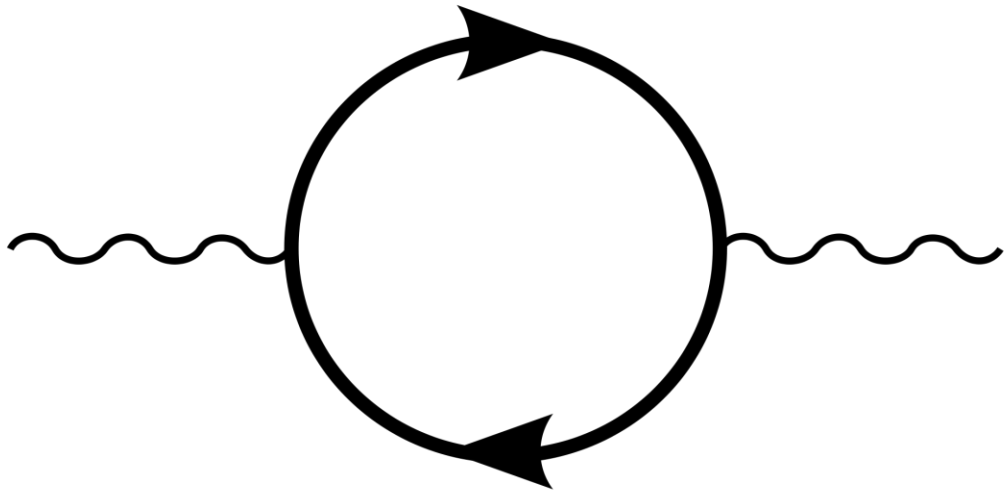
- Introduce constraint lambda
- Integrate out n
- Expand to second order around lambda = A_mu = 0;
- gauge invariance requires A^2 term to look like F_munu^2
- Find coefficient N/(48piM^2)

$$S^{(2)}_{\text{eff}} = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2} A_\mu(-p)\Pi_{\mu\nu}(p)A_\nu(p)$$

$$= \frac{N}{48\pi M^2}\int \mathrm{tr}\,F_{\mu\nu}F_{\mu\nu}$$

Large N expansion

- Effective theory looks like effective QED!
- Theta term of same form corresponds to **background E field**, strength parameterized by theta
- F does not have periodic minima
- Expansion is in orders of (1/N); **instantons barely contribute**



$$F(\theta)/L = \frac{3}{2\pi N}(M\theta)^2 + O(N^{-2})$$

T>0: Effective Mass

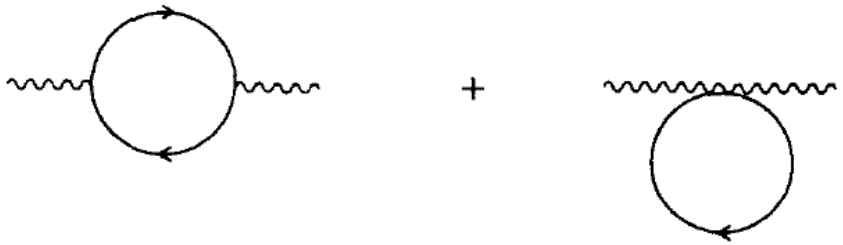
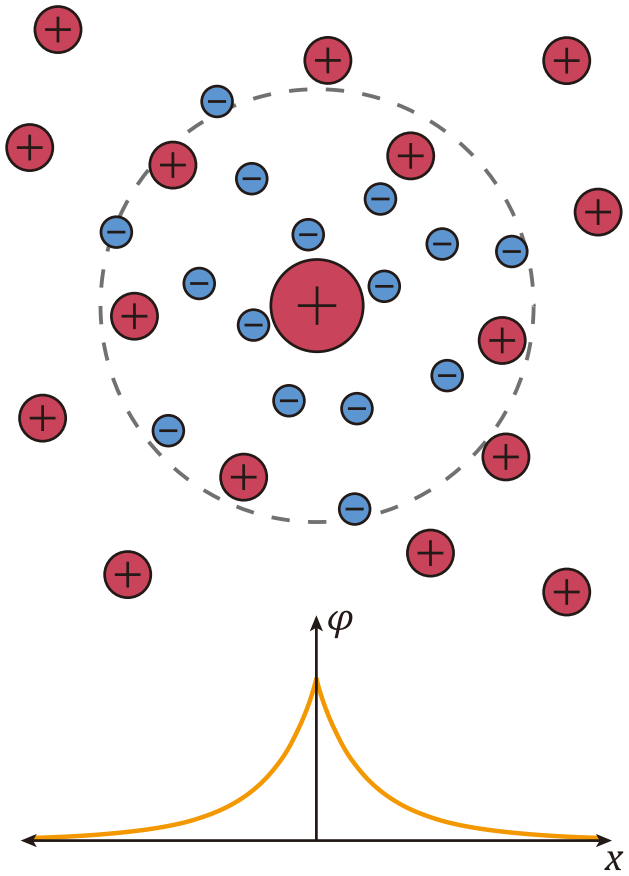


Fig. 1. The one-loop vacuum polarization graphs due to a massive charged scalar.

- Lorentz invariance is BROKEN; zero mass is not necessarily enforced.
- Specialize to A_1=0 gauge; Pi_{00}(0) does not necessarily = 0
- Compute loop contributions; get photon mass greater than zero…
- KEY: computations now require **SUMMATIONS over k1**



T>0 and summation

- Effective action becomes ->

$$S_{\text{eff}} = N \operatorname{tr} \ln(-D^2 + M^2)$$
$$= \int_0^\beta d\tau \int dx \left[\frac{N}{\beta} \sum_n \int \frac{k_1}{2\pi} \ln\left((k_0 + A_0)^2 + k_1^2 + M^2 \right) \right]$$

- Get sum from poles of cotangent function in contour integral;

$$I(\beta) = \frac{1}{\beta} \sum_n \int \frac{dk_1}{2\pi} [(\frac{2\pi n}{\beta})^2 + k_1^2 + M^2]^{-1}$$
$$= \int \frac{dk_1}{2\pi} \frac{1}{4\pi i} \oint dz \frac{\cot(\beta z/2)}{z^2 + \omega^2}$$

- Shifting contours, computing residues and spamming inverse trig identities, we get a decently nice result

$$= \int \frac{dk_1}{2\pi} \frac{1}{\omega} \frac{\sinh(\beta\omega)}{\cosh(\beta\omega) - \cosh(\beta A_0)}$$

Manipulations of potential

- Further manipulating the action, we can write it as an integral over a potential

$$V(A_0) = \frac{N}{\beta} \int \frac{dk_1}{2\pi} \ln \left(1 + \frac{\sin^2(\beta A_0/2)}{\sinh^2(\beta(k_1^2 + M^2)/2)} \right)$$

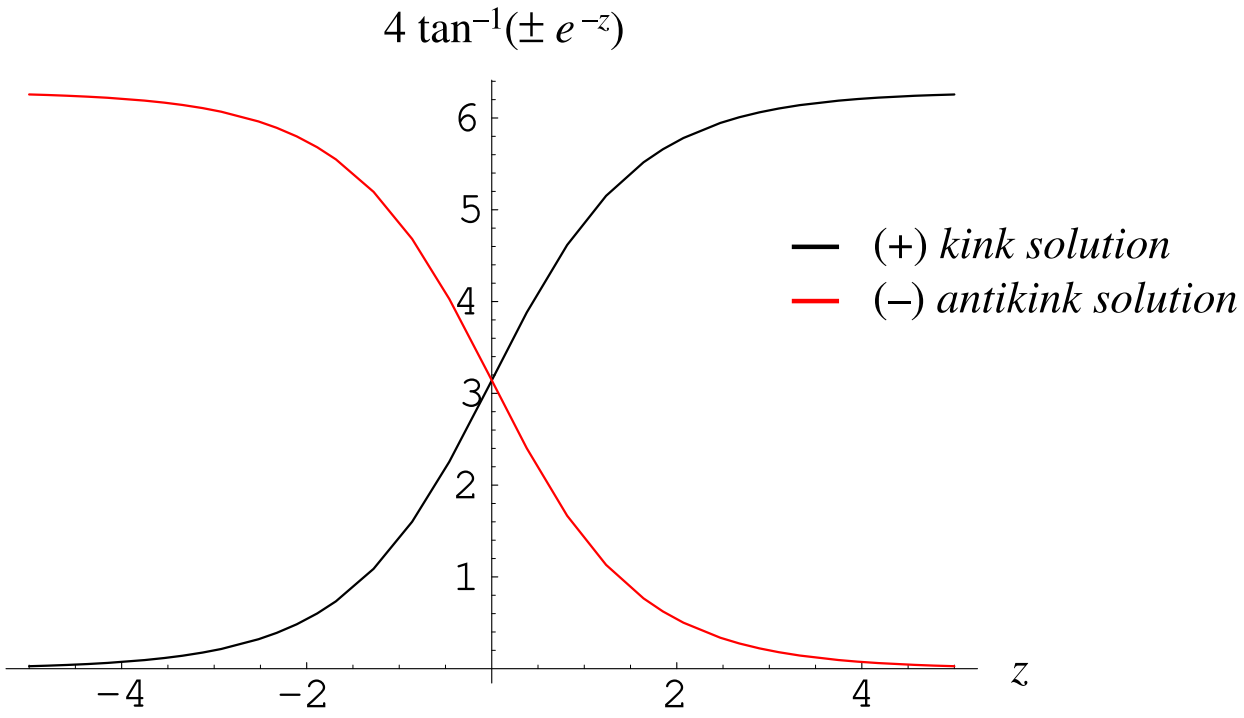
- Requiring finite action, we **recover topological behavior**!

$$A_0(x \to \pm\infty) \to 2\pi n_\pm / \beta,$$

- Action becomes **sine-Gordo**n for \beta M >> 1:

$$S_{\text{eff}}[A_0] = N \int d\tau \int dx_1 \left[\frac{1}{24\pi M^2} (\partial_1 A_0)^2 + \frac{\xi}{\beta} (1 - \cos(\beta A_0)) \right]$$

T>0 gives calorons!



- Instantons are 1D kinks
- Recover topological charge

$$Q = \frac{\beta}{2\pi}(A_0(\infty) - A_0(-\infty)) = 1 - 0 = 1$$

- Can be shown to solve \lambda constraint as well

$$S_0 = \frac{2N}{M}\sqrt{\frac{\xi}{3\pi\beta}}$$

Calorons produce theta dependence

- Decompose partition function into topological sectors
- Expand as dilute caloron gas (in right T range, contributions dominate)
- Solve for free energy! Cosine indicates importance of instantons.

$$Z(\theta) = \sum_{Q} e^{iQ\theta} Z_Q$$

$$Z(\theta) = \sum_{n_\pm} \frac{1}{n_+! n_-!} (KL\beta e^{S_0 + i\theta})^{n_+} (KL\beta e^{-S_0 - i\theta})^{n_-}$$

$$\frac{F(\theta)}{L} := -2K \exp\left[-\frac{2N}{M} \left(\frac{\xi}{3\pi\beta} \right)^{1/2} \right] \cos(\theta)$$

Analogy to QED

- Effective potential is equivalent to low order! Change coupling constant.

$$S_{\mathrm{QED}} \approx \int_0^\beta d\tau \int dx_1 \left[\frac{1}{2e^2} (\partial_1 A_0)^2 + \frac{\xi}{\beta} (1 - \cos(\beta A_0)) \right]$$

- Sine-Gordon behavior. Can derive from this a classical coulomb gas

$$= \sum_{n_\pm = 0}^{\infty} \frac{(\xi/2)^{n_+ + n_-}}{n_+! n_-!} \left(\Pi_i^{n_+} \int dx_i^+ \right) \left(\Pi_j^{n_-} \int dx_j^- \right) \exp \left[-\frac{\beta e^2}{4} \int dx dy \, \rho(x) |x - y| \rho(y) \right]$$

Implications for QCD

- QCD has similar behavior to CP^{N-1} model
- At T=0, QCD theta term comes from perturbative expansion
- At T>0, instanton effects dominate, resulting in cosine-like free energy
- Affleck speculates generation by similar calorons;

This real culprit: monopoles! Behavior, however, is similar.

Citations

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