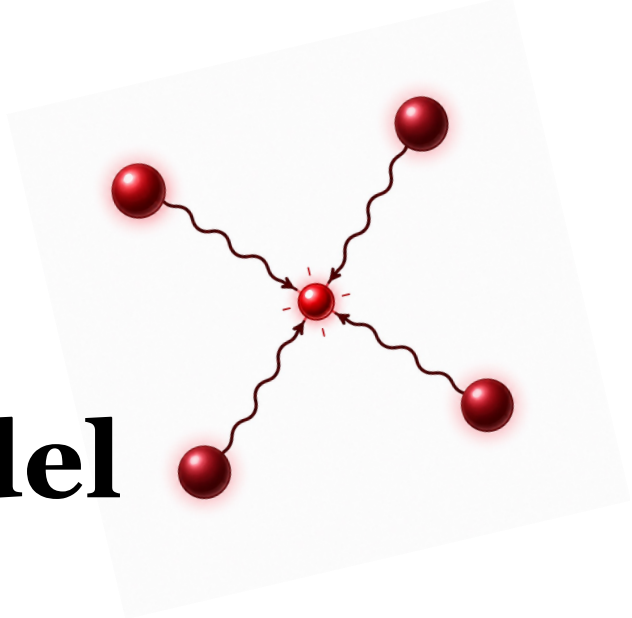
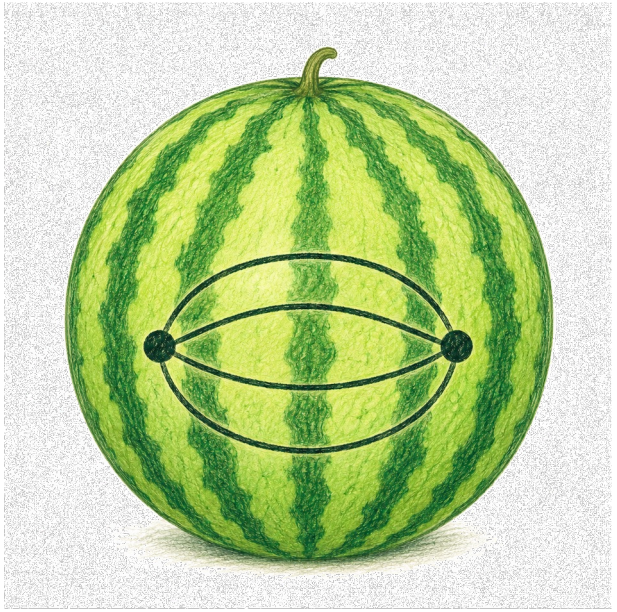
PH 332 · QFT III | FINAL PRESENTATION

Large-N Solution of the Sachdev–Ye–Kitaev Model

Melon Diagrams and Emergent Conformal Symmetry





Parry Luo

Department of Applied Physics, Stanford University

Jun. 1, 2026

Stanford University

OUTLINE

1 Introduction
Motivation
Historical origins

2 SYK Basics
The Hamiltonian
Disorder contraction

3 Effective Action
Majorana path integral
Disorder averaging
Introducing bilocal fields
Integrating out fermions
Saddle point equations

4 Diagrammatics
Perturbative expansion & Feynman rules
Index counting & melon diagrams
Resummation to Schwinger-Dyson equations

5 IR Limit
The low-energy/strong-coupling limit
Power law ansatz: $\Delta = 1/4$
Reparametrization symmetry

6 Schwarzian Action
Short-distance expansion
Symmetry-breaking pattern

7 Summary

SECTION I: INTRODUCTION

Motivation

Historical origins

Motivation: Why SYK Model?

*A strongly-coupled many-body system that is **chaotic**, yet exactly **solvable** at large N and in the IR — a controlled bridge between three fields.*

Condensed Matter

Non-Fermi liquid / strange metal

No quasiparticles; Planckian dissipation

$$\frac{1}{\tau} \sim \frac{k_B T}{\hbar}$$

High-Energy / Gravity

Holographically dual to 2D dilaton (JT) gravity

near-extremal black-hole throats

Quantum Information

Maximally chaotic

saturates the MSS bound

$$\lambda_L = \frac{2\pi}{\beta}$$

, fastest scrambler.

Historical Origins

The Sachdev–Ye Model: A random quantum magnet (1993)

Sachdev, S. & Ye, J *Physical Review Letters* **70**, 3339-3342 (1993)

$$H_{\mathrm{SY}} = \frac{1}{\sqrt{NM}} \sum_{i<j} J_{ij}\ \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j, \qquad \overline{J_{ij}} = 0, \quad \overline{J_{ij}^2} = J^2$$

$$\tilde{\chi}(\omega) = X\left(\ln\frac{1}{|\omega|} + i\frac{\pi}{2}\mathrm{sgn}(\omega)\right) + \cdots$$

- N spins with SU(M) symmetry, all-to-all Gaussian exchange.
- Solved via Schwinger bosons + disorder average + large-M saddle.

Kitaev's Reformulation (2015)

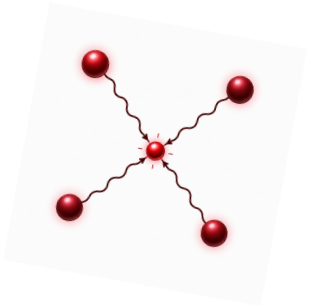
Kitaev, A simple model of quantum holography, (2015)

Majorana fermions

No constraint, no chemical potential.

Four-body interaction

Simplest interaction not reducible to free fermions → non-Fermi liquid.



$$H_{\mathrm{SYK}} = \sum_{1\le i<j<k<l\le N} J_{ijkl}\,\chi_i\chi_j\chi_k\chi_l = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\,\chi_i\chi_j\chi_k\chi_l$$

The SYK Hamiltonian

$$H_{\text{SYK}} = \sum_{1\le i<j<k<l\le N} J_{ijkl}\,\chi_i\chi_j\chi_k\chi_l = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\,\chi_i\chi_j\chi_k\chi_l$$

Random coupling tensor J_{ijkl}

$$P(J_{ijkl}) \propto \exp\left[-\frac{N^3}{12J^2}J_{ijkl}^2\right]$$

- Mean $\quad \overline{J_{ijkl}} = 0$
- Variance $\quad \overline{J_{ijkl}^2} = \frac{3!\,J^2}{N^3}$

Since Majorana operators anticommute,

$$J_{iikl} = -J_{iikl} = 0$$

$$J_{\sigma(i)\sigma(j)\sigma(k)\sigma(l)} = \text{sgn}(\sigma) J_{ijkl}$$

Number of independent couplings:

$$\binom{N}{4} = \frac{N!}{4!\,(N-4)!}$$

Disorder Contraction

$$\overline{J_{i_1 i_2 i_3 i_4} J_{j_1 j_2 j_3 j_4} J_{k_1 k_2 k_3 k_4} J_{l_1 l_2 l_3 l_4}}$$

$$= \overline{J_{i_1 i_2 i_3 i_4} J_{j_1 j_2 j_3 j_4}} \cdot \overline{J_{k_1 k_2 k_3 k_4} J_{l_1 l_2 l_3 l_4}} + \overline{J_{i_1 i_2 i_3 i_4} J_{k_1 k_2 k_3 k_4}} \cdot \overline{J_{j_1 j_2 j_3 j_4} J_{l_1 l_2 l_3 l_4}} + \overline{J_{i_1 i_2 i_3 i_4} J_{l_1 l_2 l_3 l_4}} \cdot \overline{J_{j_1 j_2 j_3 j_4} J_{k_1 k_2 k_3 k_4}}$$

Disorder average of two arbitrary couplings

$$\overline{J_{i_1 i_2 i_3 i_4} J_{j_1 j_2 j_3 j_4}} = \frac{3! J^2}{N^3} \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \delta_{i_1 j_{\sigma(1)}} \delta_{i_2 j_{\sigma(2)}} \delta_{i_3 j_{\sigma(3)}} \delta_{i_4 j_{\sigma(4)}}$$

An important special case:

$$\sum_{k,l,m=1}^{N} \overline{J_{iklm} J_{jklm}} = 3! J^2 \delta_{ij} + \mathcal{O}\left(\frac{J^2}{N}\right)$$

Three free internal sums (k,l,m) each give a factor N, cancelling the N^{-3} from the variance ⟹ the contraction is O(1).

SECTION III: EFFECTIVE ACTION

Majorana path integral

Disorder averaging

Auxiliary bilocal fields

Saddle point equations

Majorana Path Integral

The partition function (in the Euclidean time $\tau = it$)

$$Z = \mathrm{Tr} e^{-\beta H} = \int \mathcal{D}\chi(\tau)\, e^{-I[\chi]}$$

The SYK action is

$$I_{\mathrm{SYK}} = \int_0^\beta d\,\tau \left[\frac{1}{2}\sum_i \chi_i\, \partial_\tau \chi_i - \sum_{i<j<k<l} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l\right]$$

- **Free Majorana propagators**

Zero temperature

$$G_0^{-1}(\tau,\tau') = \delta(\tau-\tau')\partial_\tau \qquad G_0(\tau,\tau') = \frac{1}{2}\mathrm{sgn}(\tau-\tau')$$

Finite temperature

Anti-periodic boundary condition

$$\chi(\tau+\beta) = -\chi(\tau).$$

$$G_0^\beta(\tau) = \frac{1}{2}\mathrm{sgn}\left[\sin\left(\frac{\pi\tau}{\beta}\right)\right]$$

Disorder Averaging

Replica trick

$$\overline{\log Z} = \lim_{M\to 0} \left(\overline{Z^M} - 1\right)/M \qquad \overline{Z^M} = \overline{Z}^M + O\left(\frac{1}{N}\right) \implies \overline{\log Z} \approx \log \overline{Z}$$

The fluctuations of Z are suppressed by powers of 1/N, the replica trick is unnecessary and we can compute $\overline{Z}$ directly

Average over J_{ijkl}

$$\overline{f(J_{ijkl})} = \int \mathcal{D}J_{ijkl}\, f(J_{ijkl})$$

$$\mathcal{D}J_{ijkl} = \prod_{i<j<k<l} \frac{1}{\sqrt{2\pi \frac{3!J^2}{N^3}}} \exp\left[-\frac{1}{2\frac{3!J^2}{N^3}} J_{ijkl}^2\right] dJ_{ijkl}$$

$$= \left(\prod_{i<j<k<l} \sqrt{\frac{N^3}{3!J^2}}\, \frac{dJ_{ijkl}}{\sqrt{2\pi}}\right) \exp\left[-\frac{N^3}{12J^2} \sum_{i<j<k<l} J_{ijkl}^2\right]$$

The partition function becomes

$$\overline{Z} = \int \mathcal{D}J_{ijkl}\, \mathcal{D}\chi_i \exp\left\{-\int d\tau \left[\frac{1}{2}\sum_i \chi_i\, \partial_\tau \chi_i - \frac{1}{4!}\sum_{i,j,k,l} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l\right]\right\}$$

$$= \int \mathcal{D}\chi_i \exp\left[-\int d\tau\, \frac{1}{2}\sum_i \chi_i\, \partial_\tau \chi_i\right] \int \mathcal{D}J_{ijkl} \exp\left[\int d\tau \sum_{i<j<k<l} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l\right]$$

Disorder Averaging

$$\int \mathcal{D}J_{ijkl} \exp\left[\int d\tau \sum_{i<j<k<l} J_{ijkl}\,\chi_i\chi_j\chi_k\chi_l\right] = \exp\left[\frac{1}{2}\frac{3!\,J^2}{N^3}\sum_{i<j<k<l}\left(\int d\tau\,\chi_i\chi_j\chi_k\chi_l\right)^2\right]$$

Gaussian integral identity

$$\int dx \exp\left[-\frac{x^2}{2\sigma^2} - xA\right] = \int dx \exp\left[-\frac{1}{2\sigma^2}(x^2 + 2\sigma^2 Ax + \sigma^4 A^2) + \frac{\sigma^2 A^2}{2}\right]$$

$$= \sqrt{2\pi\sigma^2}\exp\left[\frac{\sigma^2 A^2}{2}\right]$$

Let $\sigma = \sqrt{\frac{3!\,J^2}{N^3}}$, $\quad A_{ijkl} = \int d\tau\,\chi_i\chi_j\chi_k\chi_l$

Disorder Averaging

$$\int \mathcal{D}J_{ijkl} \exp\left[\int d\tau \sum_{i<j<k<l} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l\right] = \exp\left[\frac{1}{2}\frac{3!\,J^2}{N^3} \sum_{i<j<k<l} \left(\int d\tau\, \chi_i \chi_j \chi_k \chi_l\right)^2\right]$$

Now, the partition function becomes

$$\overline{Z} = \int D\chi \exp\left[-\frac{1}{2}\int d\tau \sum_i \chi_i(\tau)\, \partial_\tau \chi_i(\tau) + \frac{3!\,J^2}{2N^3}\frac{1}{4!} \sum_{i,j,k,l} \left(\int d\tau\, \chi_i \chi_j \chi_k \chi_l\right)^2\right]$$

Using

$$\sum_{i,j,k,l} \left(\int d\tau\, \chi_i \chi_j \chi_k \chi_l\right)^2 = N^4 \int d\tau d\tau' \left[\frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau')\right]^4$$

we have

$$\overline{Z} = \int D\chi \exp\left[-\frac{1}{2}\int d\tau \sum_i \chi_i\, \partial_\tau \chi_i + \frac{NJ^2}{8}\int d\tau d\tau'\, \Xi(\tau,\tau')^4\right] \qquad \Xi(\tau,\tau') = \frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau')$$

Auxiliary Bilocal Fields Σ and G

$$\overline{Z} = \int D\chi \exp\left[-\frac{1}{2}\int d\tau \sum_i \chi_i \,\partial_\tau \chi_i + \frac{NJ^2}{8}\int d\tau d\tau' \,\Xi(\tau,\tau')^4\right] \qquad \Xi(\tau,\tau') = \frac{1}{N}\sum_i \chi_i\,(\tau)\chi_i(\tau')$$

We now introduce an independent bilocal field $G\ (\tau,\ \tau')$ by inserting a functional delta function

$$f(\xi) = \int dx\, f(x)\delta(x-\xi) \qquad \delta(x-\xi) = \frac{N}{4\pi}\int dy \exp\left[\frac{iN}{2}(x-\xi)y\right].$$

Let

$$x = G(\tau,\tau'), \qquad y = i\Sigma(\tau,\tau'), \qquad \xi = \Xi(\tau,\tau')$$

with the normalization condition

$$1 = \int DG\, D\Sigma \exp\left[-\frac{N}{2}\int d\tau d\tau' \left(G(\tau,\tau') - \Xi(\tau,\tau')\right)\Sigma(\tau,\tau')\right]$$

Auxiliary Bilocal Fields Σ and G

Let

$$x = G(\tau,\tau'), \qquad y = i\Sigma(\tau,\tau'), \qquad \xi = \Xi(\tau,\tau')$$

with the normalization condition

$$1 = \int DG\, D\Sigma \exp\left[-\frac{N}{2}\int d\tau d\tau' \left(G(\tau,\tau') - \Xi(\tau,\tau')\right)\Sigma(\tau,\tau')\right]$$

The averaged partition function becomes

$$\overline{Z} = \int D\chi \exp\left[\int d\tau d\tau' \left(-\frac{N}{2}G_0^{-1}(\tau,\tau')\Xi(\tau,\tau') + \frac{NJ^2}{8}\Xi(\tau,\tau')^4\right)\right]$$

$$= \int DG\, D\Sigma\, D\chi \exp\left[\int d\tau d\tau' \left(-\frac{N}{2}G_0^{-1}(\tau,\tau')\Xi(\tau,\tau') + \frac{N}{2}\Sigma(\tau,\tau')\Xi(\tau,\tau')\right)\right]$$

$$\times \exp\left[\int d\tau d\tau' \frac{N}{2}\left(\frac{J^2}{4}G(\tau,\tau')^4 - G(\tau,\tau')\Sigma(\tau,\tau')\right)\right]$$

Auxiliary Bilocal Fields Σ and G

Let

$$x = G(\tau,\tau'), \qquad y = i\Sigma(\tau,\tau'), \qquad \xi = \Xi(\tau,\tau')$$

with the normalization condition

$$1 = \int DG\, D\Sigma \exp\left[-\frac{N}{2}\int d\tau d\tau' \left(G(\tau,\tau') - \Xi(\tau,\tau')\right)\Sigma(\tau,\tau')\right]$$

The averaged partition function becomes

$$\overline{Z} = \int D\chi \exp\left[\int d\tau d\tau' \left(-\frac{N}{2}G_0^{-1}(\tau,\tau')\Xi(\tau,\tau') + \frac{NJ^2}{8}\Xi(\tau,\tau')^4\right)\right]$$

$$= \int DG\, D\Sigma\, D\chi \exp\left[\int d\tau d\tau' \left(-\frac{N}{2}G_0^{-1}(\tau,\tau')\Xi(\tau,\tau') + \frac{N}{2}\Sigma(\tau,\tau')\Xi(\tau,\tau')\right)\right]$$

$$\times \exp\left[\int d\tau d\tau' \frac{N}{2}\left(\frac{J^2}{4}G(\tau,\tau')^4 - G(\tau,\tau')\Sigma(\tau,\tau')\right)\right]$$

Integrating out Majorana Fermions

The fermionic exponent is a quadratic term

$$\int D\chi \exp\left[-\frac{1}{2}\int d\tau d\tau' \sum_i \chi_i(\tau)[\delta(\tau-\tau')\partial_\tau - \Sigma(\tau,\tau')]\chi_i(\tau')\right]$$

$$= \int D\chi_i \exp\left[-\frac{1}{2}\int d\tau d\tau' \chi_i(\tau)K(\tau,\tau')\chi_i(\tau')\right] \qquad K(\tau,\tau') = G_0^{-1}(\tau,\tau') - \Sigma(\tau,\tau').$$

Integrating over N Majorana fermions, we have

$$\int \prod_{i=1}^{N} D\chi_i \exp\left[-\frac{1}{2}\int d\tau d\tau' \chi_i(\tau)K(\tau,\tau')\chi_i(\tau')\right] = \mathrm{Pf}(K) \qquad \mathrm{Pf}(K)^2 = \det K$$

$$\int \prod_{i=1}^{N} D\chi_i \exp\left[-\frac{1}{2}\int d\tau d\tau' \sum_{i=1}^{N} \chi_i(\tau)K(\tau,\tau')\chi_i(\tau')\right] = [\mathrm{Pf}(K)]^N = [\det K]^{N/2} = \exp\left[\frac{N}{2}\mathrm{Trlog}K\right]$$

Now, the partition function becomes

$$\overline{Z} = \int DG\, D\Sigma \exp\left[\frac{N}{2}\mathrm{Trlog}(G_0^{-1} - \Sigma)\right] \exp\left[\frac{N}{2}\int d\tau d\tau' \left(\frac{J^2}{4}G(\tau,\tau')^4 - \Sigma(\tau,\tau')G(\tau,\tau')\right)\right]$$

$$= \int DG\, D\Sigma\, e^{-I_{\mathrm{eff}}[G,\Sigma]}.$$

Finding the Saddle Point

The effective action is

$$\frac{I_{\text{eff}}[G,\Sigma]}{N} = -\frac{1}{2}\text{Trlog}(G_0^{-1} - \Sigma) + \frac{1}{2}\int d\tau d\tau' \left(\Sigma G - \frac{J^2}{4}G^4\right)$$

Variation with respect to G

$$\frac{\delta_G I_{\text{eff}}}{\delta G} = 0 \qquad \Rightarrow \qquad \Sigma(\tau,\tau') = J^2 G^3(\tau,\tau').$$

Variation with respect to Σ

$$\frac{\delta_\Sigma I_{\text{eff}}}{\delta \Sigma} = 0 \qquad \Rightarrow \qquad [G_0^{-1}(\tau,\tau') - \Sigma(\tau,\tau')]^{-1} = G(\tau,\tau').$$

Schwinger-Dyson equations

$$G^{-1} = G_0^{-1} - \Sigma$$

$$\Sigma = J^2 G^3$$

SECTION IV: DIAGRAMMATICS

Perturbative Eexpansion

Melon Diagrams

Resummation to Schwinger-Dyson Equations

Perturbative Expansion

$$U(\tau_2,\tau_1) = \mathcal{T}\exp\left[-\int_{\tau_1}^{\tau_2} H_I(\tau)\,d\tau\right]$$

$$= 1 - \int_{\tau_1}^{\tau_2} H_I(\tau)\,d\tau + \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau \int_{\tau_1}^{\tau_2} d\tau'\,\mathcal{T}[H_I(\tau)H_I(\tau')] + \cdots$$

$$= 1 - \int_{\tau_1}^{\tau_2} H_I(\tau)\,d\tau + \int_{\tau_1}^{\tau_2} d\tau \int_{\tau_1}^{\tau} d\tau'\,H_I(\tau)H_I(\tau') + \cdots$$

$$G(\tau)\delta_{ab} = \langle \mathcal{T}[\chi_a(\tau)\chi_b(0) - \frac{1}{4!}\sum_{i,j,k,l}\overline{J_{ijkl}}\int_{-\infty}^{\infty} d\tau'\,\chi_a(\tau)\chi_b(0)\chi_i'\chi_j'\chi_k'\chi_l'$$

$$+\frac{1}{2}\left(\frac{1}{4!}\right)^2\sum_{i,j,k,l}\sum_{p,q,r,s}\overline{J_{ijkl}J_{pqrs}}\int_{-\infty}^{\infty} d\tau'\int_{-\infty}^{\infty} d\tau''\,\chi_a(\tau)\chi_b(0)\chi_i'\chi_j'\chi_k'\chi_l'\chi_p''\chi_q''\chi_r''\chi_s'' + \mathcal{O}(J^3)]\rangle$$

Perturbative Expansion

$$\overline{J_{ijkl}} = 0$$

$$G(\tau)\delta_{ab} = \langle \mathcal{T}[\chi_a(\tau)\chi_b(0) - \frac{1}{4!}\sum_{i,j,k,l}\overline{J_{ijkl}}\int_{-\infty}^{\infty} d\tau'\,\chi_a(\tau)\chi_b(0)\chi_i'\chi_j'\chi_k'\chi_l'$$

$$+\frac{1}{2}\left(\frac{1}{4!}\right)^2\sum_{i,j,k,l}\sum_{p,q,r,s}\overline{J_{ijkl}J_{pqrs}}\int_{-\infty}^{\infty} d\tau'\int_{-\infty}^{\infty} d\tau''\,\chi_a(\tau)\chi_b(0)\chi_i'\chi_j'\chi_k'\chi_l'\chi_p''\chi_q''\chi_r''\chi_s'' + \mathcal{O}(J^3)]\rangle$$

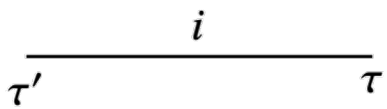
Perturbative Expansion

$$G(\tau)\delta_{ab} = \langle \mathcal{T}\chi_a(\tau)\chi_b(0)\rangle$$

$$+\left\langle \frac{1}{2}\left(\frac{1}{4!}\right)^2 \sum_{i,j,k,l,p,q,r,s} \overline{J_{ijkl}J_{pqrs}} \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' \mathcal{T}\chi_a(\tau)\chi_b(0)\chi_i'\chi_j'\chi_k'\chi_l'\chi_p''\chi_q''\chi_r''\chi_s'' \right\rangle + \mathcal{O}(J^4)$$

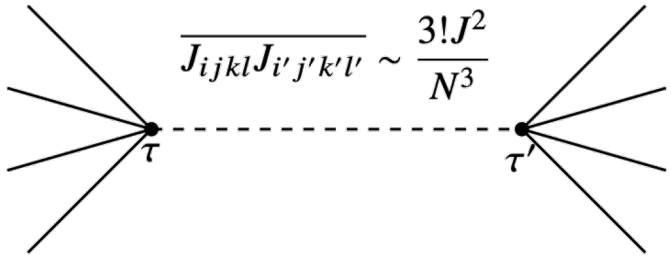
Feynman rules

Propagators



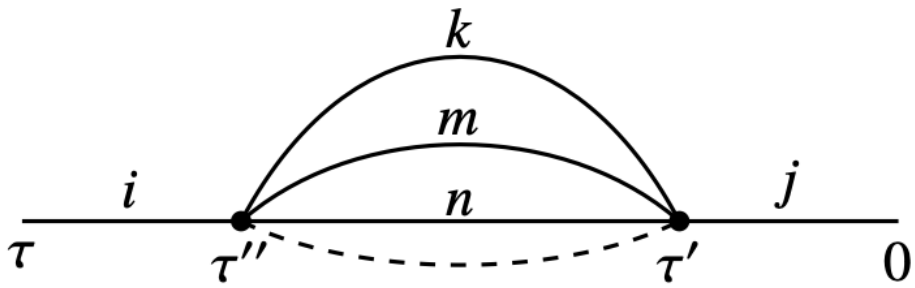
$$G_0(\tau,\tau') = \frac{1}{2}\mathrm{sgn}(\tau-\tau')$$

Vertices



Melon Diagrams

The leading correction (second order in J)



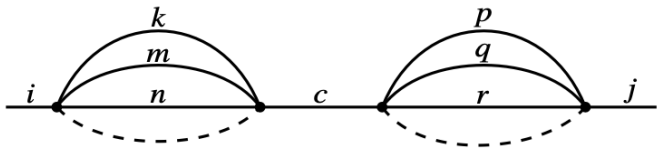
$$\sim \frac{J^2}{N^3} \times N^3 = J^2 = O(1)$$

$$G(\tau) - G_0(\tau) = \frac{2 \cdot 4 \cdot 4!}{2 \cdot 4! \cdot 4!} \sum_{i,j,k,m,n} \overline{J_{ikmn} J_{jkmn}} \int d\tau'\, d\tau''\, G_0(\tau - \tau'') \delta_{ia} G_0(\tau'' - \tau')^3 G_0(\tau') \delta_{jb}$$

$$+ O\left(\frac{J^2}{N}\right) + O(J^4)$$

Melon vs. Non-melon Diagrams

Melon

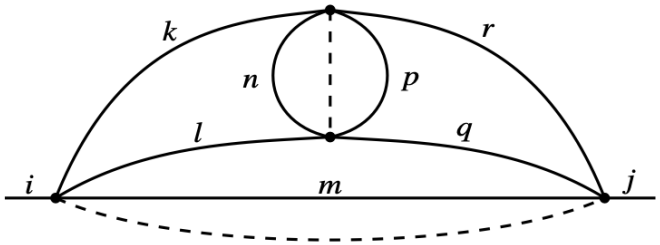


$$\overline{J_{ikmn}J_{ckmn}} \sim \frac{3!J^2}{N^3}\delta_{ic}\delta_{kk}\delta_{mm}\delta_{nn},$$

$$\overline{J_{cpqr}J_{jpqr}} \sim \frac{3!J^2}{N^3}\delta_{cj}\delta_{pp}\delta_{qq}\delta_{rr}.$$

$$\overline{J_{ikmn}J_{ckmn}}\,\overline{J_{cpqr}J_{jpqr}} \sim \frac{J^2}{N^3}\times\frac{J^2}{N^3}\times N^6 = J^4 \sim \mathcal{O}(1)$$

Non-melon



$$\overline{J_{iklm}J_{jqrm}} \sim \frac{3!J^2}{N^3}\delta_{ij}\delta_{kq}\delta_{lr}\delta_{mm},$$

$$\overline{J_{krnp}J_{qlnp}} \sim \frac{3!J^2}{N^3}\delta_{kq}\delta_{rl}\delta_{nn}\delta_{pp}.$$

$$\overline{J_{iklm}J_{jqrm}}\,\overline{J_{krnp}J_{qlnp}} \sim \frac{J^2}{N^3}\times\frac{J^2}{N^3}\times N^5 = \frac{J^4}{N} \sim \mathcal{O}\left(\frac{1}{N}\right)$$

Summing Melon Diagrams (at large-N)

From the melon diagram, we identify the self-energy

$$\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^3$$

Each of the three internal lines in the melon diagram can itself receive melonic corrections



$$G = G_0 + G_0\,\Sigma\,G_0 + G_0\,\Sigma\,G_0\,\Sigma\,G_0 + \cdots$$

$$= G_0 + G_0\,\Sigma\,(G_0 + G_0\,\Sigma\,G_0 + \cdots)$$

$$= G_0 + G_0\,\Sigma\,G$$

$$G^{-1} = G_0^{-1} - \Sigma$$

SECTION V: IR LIMIT

Low-energy and strong-coupling regime

Power-law ansatz and conformal symmetry

Reparametrization symmetry

Low-Energy & Strong-Coupling Regime

In Fourier space,

$$G^{-1}(\omega) = G_0^{-1}(\omega) - \Sigma(\omega).$$

$$G_0(\omega) = \int_{-\infty}^{\infty} d\,\tau\, e^{i\omega\tau}\,\frac{1}{2}\,\mathrm{sgn}(\tau) = \frac{i}{\omega + i0} \qquad\qquad G_0^{-1}(\omega) = -i\omega$$

In the low energy ($\omega \ll J$) and strong coupling regime ($\beta J \gg 1$)

$$\frac{1}{J} \ll |\tau| \ll \beta,$$

$$G^{-1}(\omega) \simeq -\Sigma(\omega), \qquad \Sigma(\omega)G(\omega) \simeq -1.$$

In the time domain,

$$\int d\tau\, \Sigma(\tau_1, \tau) G(\tau, \tau_2) = -\delta(\tau_1 - \tau_2).$$

Power-law Ansatz & Conformal Symmetry

We seek a scale-invariant equation ⟹ a power-law (conformal) ansatz

$$G_c(\tau_1, \tau_2) = G_c(\tau) = B\,\frac{\mathrm{sgn}(\tau)}{|J\tau|^{2\Delta}}, \qquad \tau \equiv \tau_1 - \tau_2$$

Using the formula, we have

$$G_c(\omega) = B|J|^{-2\Delta} \cdot 2i\Gamma(1-2\Delta)\cos(\pi\Delta)|\omega|^{2\Delta-1}\mathrm{sgn}(\omega)$$
$$\propto |\omega|^{2\Delta-1}\mathrm{sgn}(\omega)$$

$$\Sigma_c(\omega) = \mathcal{F}\{J^2 G_c^3\}$$
$$\propto |\omega|^{6\Delta-1}\mathrm{sgn}(\omega)$$

Fourier transform of a signed power law:

$$I_D(\omega) \equiv \int_{-\infty}^{\infty} d\,\tau\, e^{i\omega\tau}\frac{\mathrm{sgn}(\tau)}{|\tau|^{2D}}$$
$$= 2i\Gamma(1-2D)\cos(\pi D)\mathrm{sgn}(\omega)|\omega|^{2D-1}$$

$G^{-1}(\omega) = -\Sigma(\omega)$ requires the ω-dependence of both sides to match

$$|\omega|^{1-2\Delta} \stackrel{!}{=} |\omega|^{6\Delta-1} \qquad 1-2\Delta = 6\Delta - 1 \quad \Rightarrow$$

$$\Delta = \frac{1}{4}$$

Power-law Ansatz & Conformal Symmetry

We now find the conformal coefficient B

$$G_c(\omega) = B|J|^{-2\Delta} \cdot 2i\Gamma(1-2\Delta)\cos(\pi\Delta)|\omega|^{2\Delta-1}\mathrm{sgn}(\omega)$$
$$= iB|J|^{-1/2}\sqrt{2\pi}|\omega|^{-1/2}\mathrm{sgn}(\omega),$$

$$\Sigma_c(\omega) = \mathcal{F}\{J^2G^3\}$$
$$= \mathcal{F}\left\{J^2B^3|J|^{-6/4}\frac{\mathrm{sgn}(\tau)}{|\tau|^{3/2}}\right\}$$
$$= J^{1/2}B^3\left\{2i\Gamma\left(-\frac{1}{2}\right)\cos\left(\frac{3\pi}{4}\right)|\omega|^{1/2}\mathrm{sgn}(\omega)\right\}$$
$$= iB^3J^{1/2}\sqrt{8\pi}|\omega|^{1/2}\mathrm{sgn}(\omega).$$

Fourier transform of a signed power law:

$$I_D(\omega) \equiv \int_{-\infty}^{\infty} d\,\tau\, e^{i\omega\tau}\frac{\mathrm{sgn}(\tau)}{|\tau|^{2D}}$$
$$= 2i\Gamma(1-2D)\cos(\pi D)\mathrm{sgn}(\omega)|\omega|^{2D-1}$$

Therefore,

$$G_c(\omega)\Sigma_c(\omega) = -B^4\sqrt{16\pi^2} = -B^4\cdot 4\pi = -1.$$
$$B = \frac{1}{(4\pi)^{1/4}},$$

$$G_c(\tau) = \frac{1}{(4\pi)^{1/4}}\frac{\mathrm{sgn}(\tau)}{|J\tau|^{1/2}}$$

Reparametrization Symmetry

Consider a smooth, monotonic reparametrization

$$\tau \to f(\tau), \qquad f'(\tau) > 0$$

The Majorana fermions transform as

$$\chi(\tau) \to \chi_f(\tau) = [f'(\tau)]^{\Delta}\chi\big(f(\tau)\big).$$

The two-point function transforms as

$$\begin{aligned} G_f(\tau_1, \tau_2) &= \langle [f'](\tau_1)^{\Delta}\chi(f(\tau_1))[f'(\tau_2)]^{\Delta}\chi(f(\tau_2))\rangle \\ &= [f'(\tau_1)f'(\tau_2)]^{\Delta}G\big(f(\tau_1), f(\tau_2)\big). \end{aligned}$$

and the self-energy transforms as

$$\begin{aligned} \Sigma_f(\tau_1, \tau_2) &= J^2 {G_f}^3(\tau_1, \tau_2) \\ &= J^2[f'(\tau_1)f'(\tau_2)]^{3\Delta}G^3\big(f(\tau_1), f(\tau_2)\big). \end{aligned}$$

Reparametrization Symmetry

Demanding the IR equation stay ***invariant*** gives

$$\int d\tau\, \Sigma_f(\tau_1,\tau) G_f(\tau,\tau_2) = -\delta(\tau_1-\tau_2)$$

$$[f'(\tau_1)]^{3\Delta}[f'(\tau_1)]^{\Delta}[f'(\tau)]^{4\Delta-1} = f'(\tau_1)$$

$$4\Delta - 1 = 0 \Rightarrow \Delta = \frac{1}{4}$$

$$G_f(\tau_1,\tau_2) = [f'(\tau_1)f'(\tau_2)]^{\Delta} G\big(f(\tau_1), f(\tau_2)\big).$$

$$\Sigma_f(\tau_1,\tau_2) = J^2[f'(\tau_1)f'(\tau_2)]^{3\Delta} G^3\big(f(\tau_1), f(\tau_2)\big).$$

Example: mapping from zero-temperature line to finite-temperature circle

$$f(\tau) = \tan\left(\frac{\pi\tau}{\beta}\right), \qquad f'(\tau) = \frac{\pi}{\beta}\sec^2\left(\frac{\pi\tau}{\beta}\right),$$

$$G_c^{\beta}(\tau) = [f'(\tau)f'(0)]^{1/4} G_c\big(f(\tau) - f(0)\big) = \frac{\pi^{1/4}}{\sqrt{2J\beta}} \frac{\mathrm{sgn}(\tau)}{\left|\sin\left(\frac{\pi\tau}{\beta}\right)\right|^{1/2}}.$$

SECTION VI: SCHWARZIAN ACTION

Symmetry-breaking pattern

Short-distance expansion

The Schwarzian action

Symmetry-Breaking Pattern

Which symmetry is broken by the conformal solution?

$$G_c(\tau_1,\tau_2)=\frac{B}{J^{2\Delta}}\frac{\mathrm{sgn}(\tau_1-\tau_2)}{|\tau_1-\tau_2|^{2\Delta}} \qquad G_c^f(\tau_1,\tau_2)=\frac{B}{J^{2\Delta}}\frac{[f'(\tau_1)f'(\tau_2)]^{\Delta}}{|f(\tau_1)-f(\tau_2)|^{2\Delta}}\mathrm{sgn}(\tau_1-\tau_2).$$

Symmetry-Breaking Pattern

Consider the Möbius transformation, which form the SL(2,R) group:

$$f(\tau) = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1,$$

$$\frac{f'(\tau_1)f'(\tau_2)}{[f(\tau_1) - f(\tau_2)]^2} = \frac{(c\tau_1 + d)^{-2}(c\tau_2 + d)^{-2}}{(\tau_1 - \tau_2)^2(c\tau_1 + d)^{-2}(c\tau_2 + d)^{-2}} = \frac{1}{(\tau_1 - \tau_2)^2} \qquad (\Delta = 1/4)$$

We identify the SL(2,R) group as the unbroken subgroup, which leave G_c invariant

Therefore, the symmetry-breaking pattern is

$\mathrm{Diff}(\mathbb{R}) \rightarrow SL(2,\mathbb{R})$ (zero temperature).

$\mathrm{Diff}(S^1) \rightarrow SL(2,\mathbb{R})$ (finite temperature).

Short-Distance Expansion

We begin with the effective action

$$\frac{I_{\text{eff}}[G,\Sigma]}{N} = -\frac{1}{2}\text{Trlog}(G_0^{-1} - \Sigma) + \frac{1}{2}\int d\tau_1 d\tau_2 \left[\Sigma G - \frac{J^2}{4}G^4\right]$$

and shift the Lagrange multiplier

$$\Sigma \rightarrow \Sigma + G_0^{-1}$$

The effective action splits into

$$I_{\text{eff}} = I_{\text{CFT}} + I_S,$$

$$\frac{I_{\text{CFT}}}{N} = -\frac{1}{2}\text{Trlog}(-\Sigma) + \frac{1}{2}\int d\tau_1 d\tau_2 \left[\Sigma G - \frac{J^2}{4}G^4\right],$$

$$\frac{I_S}{N} = \frac{1}{2}\int d\tau_1 d\tau_2\, G_0^{-1}(\tau_1,\tau_2)G(\tau_1,\tau_2)$$

Short-Distance Expansion

$$\frac{I_S}{N} = \frac{1}{2}\int d\tau_1 d\tau_2\, G_0^{-1}(\tau_1,\tau_2)G(\tau_1,\tau_2)$$

Since G_0^{-1} includes a delta function, we expand $G(\tau_1,\tau_2)$ for small τ separation:

$$\tau_+ = \frac{\tau_1+\tau_2}{2}, \qquad s = \tau_1 - \tau_2$$

$$\frac{[f'(\tau_1)f'(\tau_2)]^{\Delta}}{[f(\tau_1)-f(\tau_2)]^{2\Delta}} = \frac{1}{s^{2\Delta}}\left[1+\frac{\Delta s^2}{4}\left(\frac{f'''}{f'}-\left(\frac{f''}{f'}\right)^2\right)-\frac{\Delta s^2}{12}\frac{f'''}{f'}+\mathcal{O}(s^4)\right]$$

$$= \frac{1}{s^{2\Delta}}\left[1+\Delta s^2\left(\frac{1}{6}\frac{f'''}{f'}-\frac{1}{4}\left(\frac{f''}{f'}\right)^2\right)+\mathcal{O}(s^4)\right]$$

$$f' = f'(\tau_+), \qquad f'' = f''(\tau_+), \qquad f''' = f'''(\tau_+).$$

Short-Distance Expansion

$$\frac{[f'(\tau_1)f'(\tau_2)]^{\Delta}}{[f(\tau_1)-f(\tau_2)]^{2\Delta}} = \frac{1}{s^{2\Delta}}\left[1+\frac{\Delta s^2}{4}\left(\frac{f'''}{f'}-\left(\frac{f''}{f'}\right)^2\right)-\frac{\Delta s^2}{12}\frac{f'''}{f'}+\mathcal{O}(s^4)\right]$$

$$= \frac{1}{s^{2\Delta}}\left[1+\Delta s^2\left(\frac{1}{6}\frac{f'''}{f'}-\frac{1}{4}\left(\frac{f''}{f'}\right)^2\right)+\mathcal{O}(s^4)\right]$$

The Schwarzian derivative is defined as

$$\{f,\tau_+\} \equiv \frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Therefore, the reparametrized saddle can be written as

$$G_c^f(\tau_1,\tau_2) = G_c(\tau_1,\tau_2)\left[1+\frac{\Delta}{6}s^2\{f,\tau_+\}+\mathcal{O}(s^4)\right].$$

The Schwarzian Action

Substituting the f-dependent part into I_S

$$\frac{I_S[G_c^f]}{N} = \frac{1}{2}\int d\tau_1 d\tau_2\, G_0^{-1}(\tau_1,\tau_2)G_c(\tau_1,\tau_2)\left[\frac{\Delta}{6}(\tau_1-\tau_2)^2\{f,\frac{\tau_1+\tau_2}{2}\}+\mathcal{O}(s^4)\right]$$

$$\approx \frac{\Delta}{12}\int d\tau_+ ds\, \delta(s)\,\partial_s\,[G_c(s)s^2]\{f(\tau_+),\tau_+\}$$

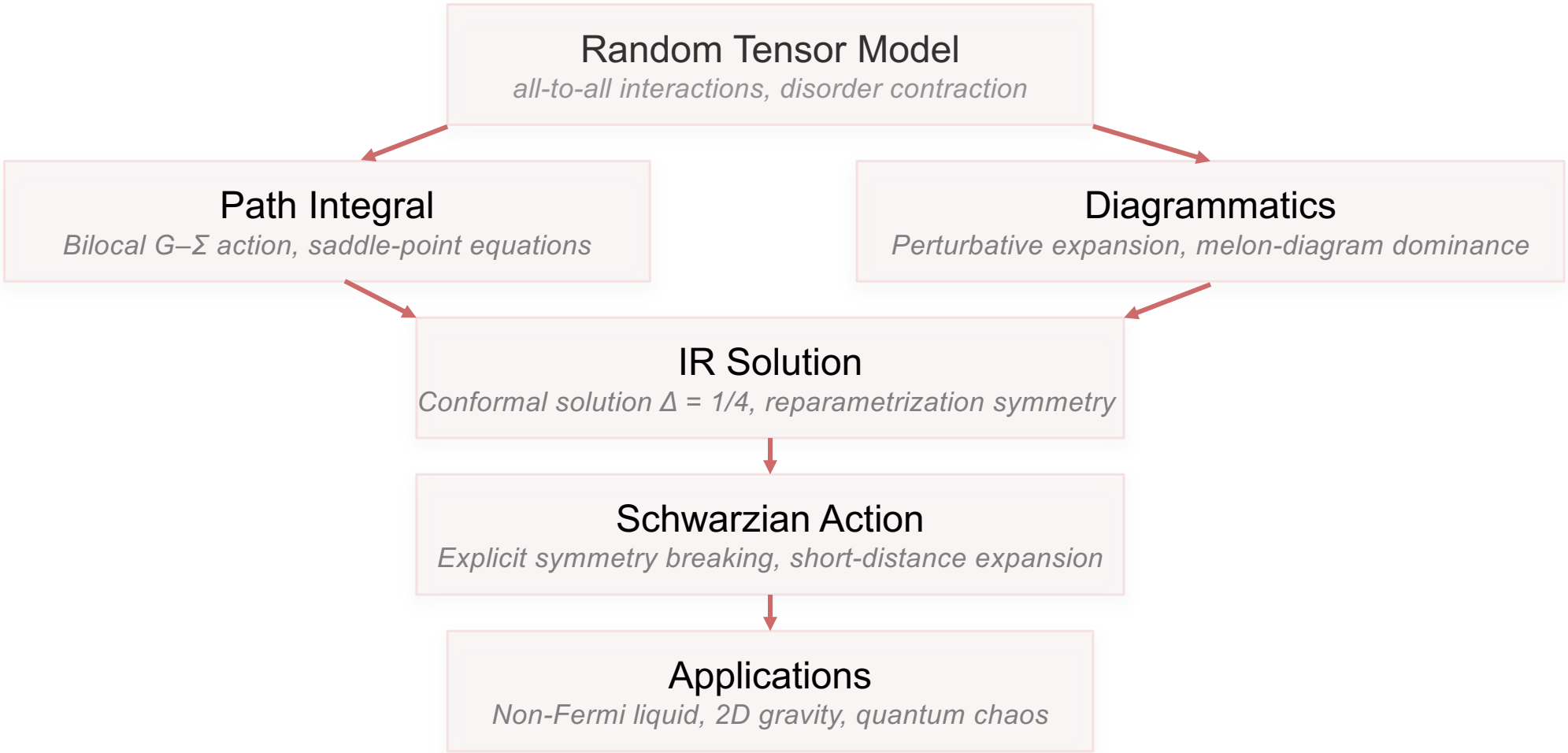
$$= \frac{\Delta}{12}\int ds\, \delta(s)\,\partial_s\,[G_c(s)s^2]\int d\tau_+\{f(\tau_+),\tau_+\}$$

$$= \frac{1}{J}\frac{\Delta}{12}\int d\eta\, \delta(\eta)\,\partial_\eta\, G_c(\eta)\eta^2\int d\tau\{f(\tau),\tau\}$$

We identify that the symmetry breaking term takes the form of

$$\frac{I_S[G_c^f]}{N} = -\frac{\alpha_S}{J}\int d\tau\,\{f(\tau),\tau\}+\cdots$$

From Random Couplings to a Universal Effective Theory



Thank You