An Introduction to Generalized Global Symmetry

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INTRODUCTION

The concept of symmetry in physics originates from the invariance of classical mechanics under spatial transformations, such as translations, rotations, and reflections. It is now an organizing principle of modern theoretical physics. In 1918, Noether published her famous theorem, which established the correspondence between continuous symmetries and conservation laws, such as the conservation of energy, momentum, and angular momentum. Later, in quantum field theory and particle physics, both global and gauge symmetries became essential for solving and formulating new theories, and they played an important role in the Standard Model.

More recently, however, studies of quantum field theory, gauge theory, and topological phases have revealed that ordinary "0-form" symmetries, which act by transforming local point-like operators, do not capture all symmetry structures. The modern framework of generalized global symmetries, introduced systematically by Gaiotto, Kapustin, Seiberg, and Willett in their seminal paper [1], extends the notion of symmetry to include higher-form symmetries acting on extended objects such as Wilson lines, surface operators, strings, and branes; these symmetries have familiar features such as Ward identities, gauging, spontaneous breaking, and 't Hooft anomalies. This perspective has been further developed to include higher-group, categorical, and non-invertible symmetries, in which symmetries may be understood through topological defects rather than ordinary group actions. Generalized symmetry has become a unifying language across high-energy theory, condensed matter physics, mathematics, string theory, and quantum gravity, with applications ranging from confinement and dualities to symmetry-protected topological phases, anomaly matching, defects, and the classification of phases of matter [2].

Despite its profound physical implications, the mathematical framework of generalized symmetry is rather abstract. In this paper, we provide a self-contained discussion of generalized global symmetry and present several examples. The paper is mainly based on explaining and deriving the results in [1–3].

MATHEMATICAL PREPARATION: MANIFOLDS AND FORMS

Differential Forms

In this section, we develop the mathematical tools related to differential forms that will be useful in this paper. Detailed treatments of differential forms can be found in many textbooks on manifolds, such as [4]. Given a manifold M, differential k-forms are completely antisymmetric covariant k-tensors, on which we may define a wedge product $\wedge$, so that for forms ω, η of degree m, n:

$$\omega \wedge \eta = \frac{(m+n)!}{m!n!}\text{alt}(\omega \otimes \eta). \tag{1}$$

Here alt is the total antisymmetrization operator. Let V be the space of 1-forms, spanned by basis vectors e^i. Then a basis for the space of k-forms is

$$e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}, \qquad i_1 < i_2 < \ldots < i_k \tag{2}$$

It is clear that for an n-dimensional space, we have $k \leq n$. As a special case, 0-forms are defined to be numbers.

If we have a metric g on V, we may define an inner product $\langle \omega, \eta \rangle$ of differential forms by setting $\langle e^I, e^J \rangle := \delta^{IJ}$ for an orthonormal basis e^i of V. It is easy to see that this definition is independent of the choice of orthonormal basis. With the help of the metric, we may also define the Hodge star operator $\star$, so that for any pair of differential forms,

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle e, \tag{3}$$

where $e^1 \wedge \ldots \wedge e^n$ is sometimes called the volume form. The volume form depends on the ordering of the orthonormal vectors e^i and changes sign under an odd permutation. The ordering of the basis vectors e^i up to even permutations is called the **orientation** of the vector space, and a basis with the correct ordering is sometimes called a right-handed basis. We will always assume that an orientation is chosen for every manifold M under consideration.

For a differential form ω, another important operation is the differential d. It is defined so that

$$d(f dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_k}. \tag{4}$$

In particular, $d^2\omega = 0$ for any differential form ω.

For a manifold M of dimension n with an orientation, top-degree differential forms ω can be integrated without referring to a metric. In a right-handed coordinate patch U,

$$\int_U \omega := \int dx\, \omega(\partial_1, \partial_2, \ldots \partial_n). \tag{5}$$

Stokes' theorem tells us that for a manifold M with boundary ∂M,

$$\int_M d\omega = \int_{\partial M} \omega. \tag{6}$$

Dual Forms for Submanifolds

In this section, we define the duality between manifolds and differential forms. For a manifold M with a metric, we can define an inner product between differential forms:

$$(\omega, \eta) := \int \langle \omega, \eta \rangle e = \int \omega \wedge \star\eta. \tag{7}$$

With this inner product, the space of forms on M is formally a Hilbert space. Given a k-dimensional submanifold Σ, the map $\eta \mapsto \int_\Sigma \eta$ is a linear functional on the Hilbert space of all forms. From our experience in quantum mechanics, we expect the existence of a differential form Θ_Σ such that

$$\int_\Sigma \eta = (\Theta_\Sigma, \eta) = \int_M \eta \wedge \star\Theta_\Sigma. \tag{8}$$

Since the $\star$ operation depends on the metric g, it is natural to define the metric-independent form $\theta_\Sigma := \star\Theta_\Sigma$, for which

$$\int_M \eta \wedge \theta_\Sigma = \int_\Sigma \eta, \tag{9}$$

which is metric independent. The form θ_Σ is called the Poincaré dual form of the manifold Σ.

As an example, consider the case where $\Sigma = \{y\}$ is a point. Then the zero-form η is a scalar function, and $\int_\Sigma \eta = \eta(y)$. The dual form θ_Σ is simply the Hodge dual of the familiar delta function:

$$\theta_{\{y\}}(x) = \delta^{(n)}(x - y)\, dx^1 \wedge \ldots \wedge dx^n \tag{10}$$

for an oriented coordinate basis. Note that both the delta function and the basis $dx^1 \wedge \ldots \wedge dx^n$ depend on coordinates, but their product does not. More generally, it is not hard to verify that for a k-dimensional submanifold Σ embedded in an n-dimensional manifold M, we have:

$$\theta_\Sigma(x) = \frac{\epsilon_{i_1 \ldots i_n}}{q!(D-q)!} \left(\int_\Sigma \delta^{(n)}(x - y) dy^{i_1} \wedge \ldots \wedge dy^{i_k} \right) dx^{i_{k+1}} \wedge \ldots \wedge dx^{i_n}, \tag{11}$$

where $\epsilon_{i_1 \ldots i_n}$ is the totally antisymmetric symbol.

With the help of Stokes' theorem, we can relate the dual forms of a submanifold V of dimension k and its boundary ∂V:

$$\int_{\partial V} \omega = \int_M \omega \wedge \theta_{\partial V} = \int_V d\omega = \int_M d\omega \wedge \theta_V . \tag{12}$$

Integrating by parts using Stokes' theorem gives:

$$\int_M \omega \wedge \theta_{\partial V} = \int_M d\omega \wedge \theta_V = (-1)^{\deg \omega + 1} \int_M \omega \wedge d\theta_V = . \tag{13}$$

We find that

$$\theta_{\partial V} = (-1)^{\dim V} d\theta_V . \tag{14}$$

In particular, V is closed/exact if and only if the dual form θ_V is closed/exact.

Linking Numbers

In the following sections, we will sometimes need the **linking number** $L(\Sigma, C)$ between two closed submanifolds Σ and C in a manifold M with $\dim \Sigma + \dim C = \dim M - 1$. Here Σ is associated with symmetry operators, and C is associated with charged operators. To discuss the linking number, we usually need to assume that Σ is contractible. The idea of the linking number is to count whether the surface Σ "encloses" C. The precise definition requires us to choose a "contraction" V such that $\partial V = \Sigma$. With the help of V, we define

$$L(\Sigma, C) := \int_V \theta_C = \int_M \theta_C \wedge \theta_V . \tag{15}$$

It is easy to see that the result is independent of continuous deformations of V, Σ, and C as long as Σ and C do not touch each other. A local consideration shows that it counts the number of times C crosses V, where crossings with the opposite orientation have weight -1.

We should remark that, since V is a contraction of Σ, the result is implicitly V-dependent when there are inequivalent ways of contracting Σ. The fact that there can be different ways of enclosing the same symmetry operator is very important. For example, it shows that charged local operators that transform nontrivially under global symmetries, generalized or not, cannot have a nonzero vacuum expectation value.

REVIEW OF ORDINARY (0-FORM) GLOBAL SYMMETRY

To introduce generalized global symmetry, we first describe a formulation of ordinary global symmetry in quantum field theory for which the generalization to higher symmetries is most natural.

Classical Theory: Noether's theorem

Starting from a classical field theory with the action functional $S[\phi(x)]$, an infinitesimal transformation of the field ϕ is given by $\phi(x) \to \phi(x) + \epsilon^a \, \delta_a \phi(x)$, where $\epsilon \to 0$ is a small constant. The infinitesimal transformation is a **symmetry** of the action if the action is invariant to linear order in ϵ under the transformation. Exponentiating the infinitesimal variation $\delta\phi$ gives the finite symmetry action $\phi \to R_g(\phi)$, where $g \in G$ is an element of the symmetry group. The infinitesimal transformation $\delta_a \phi$ is then given by the action of the corresponding Lie-algebra generators X_a:

$$R_{1+\epsilon \cdot X}(\phi) = \phi + \epsilon^a \delta_a \phi + O(\epsilon^2). \tag{16}$$

Noether's theorem asserts that, for each continuous symmetry, there is an associated conserved current J^μ. There are various ways to prove this, but the simplest way is to make the global transformation local: $\epsilon \to \epsilon(x)$, so that the transformation becomes

$$\tilde{\phi}(x) = \phi(x) + \epsilon(x) \delta\phi(x), \tag{17}$$

where for simplicity we have suppressed the Lie-algebra indices of ϵ.

Since the action $S[\phi(x)]$ is local, its variation should be a local linear functional of $\epsilon(x)$. Invariance for constant ϵ means that the functional can only depend on derivatives of ϵ, which leads to

$$\delta S = -\int_M dx \, J^\mu \partial_\mu \epsilon = \int_M dx \, \epsilon (\partial_\mu J^\mu), \tag{18}$$

where J^μ is the desired current. For physical solutions $\phi_{cl}(x)$, the equation of motion forces $\delta S = 0$, which leads to the conservation law

$$\partial_\mu J^\mu = 0 \tag{19}$$

as a consequence of the equation of motion.

Before we proceed, we note that there may be modified transformation rules for the fields such that the derivatives of $\epsilon(x)$ also influence the transformation of $\phi(x)$ in Eq. (17). In fact, if we are dealing with tensor fields under local spacetime transformations, the "natural" transformation rule involves derivatives of ϵ that account for the change of the coordinate basis under the diffeomorphism generated by $\delta x^\mu = \epsilon^\mu$. Such a transformation rule can give a different current $\tilde{J}^\mu$ in Eq. (18). However, the difference between the two currents usually involves higher-order derivatives of the fields that vanish when we impose the equations of motion. The two currents are therefore equivalent "on shell".

The defining rule Eq. (18) for the currents is simple, but the integral of scalar functions over the spacetime manifold depends on the background metric. With the help of differential forms, we can rewrite it in a more geometric way. Define the **current form** j as $j := \star J$. We then have

$$\delta S = -\int_M \omega\langle d\epsilon, J\rangle = -\int_M d\epsilon \wedge j = \int_M \epsilon\, dj, \tag{20}$$

where, with a slight abuse of notation, we have identified J^μ and J_μ. Current conservation now simply means $dj = 0$. Eq. (20) makes it straightforward to define the total charge Q on some hypersurface Σ:

$$Q(\Sigma) = \int_\Sigma j. \tag{21}$$

For two spacelike hypersurfaces at different times, $\Sigma_{1,2}$, $\Sigma_2 - \Sigma_1$ is the boundary ∂V of some spacetime region V. As a result, we have global charge conservation because $dj = 0$:

$$Q(\Sigma_2) - Q(\Sigma_1) = \int_{\Sigma_2-\Sigma_1} j = \int_{\partial V} j = \int_V dj = 0. \tag{22}$$

As we shall see, this relation has important generalizations in quantum theory.

Quantum Theory: Schwinger Dyson Equations

In this section, we deal with the quantum theory. For a local quantum field theory defined by the same action functional $S[\phi(x)]$, we have the partition function

$$Z = \int D\phi\, e^{-S[\phi]}. \tag{23}$$

Given another observable functional $F[\phi(x)]$, the expectation value $\langle F\rangle$ is defined by

$$\langle F\rangle := \frac{\int D\phi\, e^{-S[\phi]} F[\phi]}{\int D\phi\, e^{-S[\phi]}}. \tag{24}$$

For example, the two-point Green's function is defined as $G(x-y) := \langle\phi(x)\phi(y)\rangle$. In the operator formalism, this is interpreted as the time-ordered expectation value of the operator $F(\hat{\phi}(x))$. The classical equation of motion remains true in its quantum version, modulo possible ordering issues for products of operators. It tells us that under the global symmetry, the correlator should not change. However, in the quantum theory, the symmetry can tell us more information about the local correlators of operators. In the path integral, assuming the invariance of the functional measure $D\phi$ under the transformation $\tilde{\phi}(x) = \phi(x) + \epsilon(x)\delta\phi(x)$, we obtain the Schwinger-Dyson equations for $\langle F[\phi(x)]\rangle$:

$$\langle \delta S[\phi] F[\phi]\rangle = \langle \delta F[\phi]\rangle \tag{25}$$

Taking the n-point correlator functional $F = \phi(x_1)\dots\phi(x_n)$, we have [5]

$$\int_M \epsilon(x)\langle dj(x)\phi(x_1)\dots\phi(x_n)\rangle = \sum_i \epsilon(x_i)\langle\phi(x_1)\dots\delta\phi(x_i)\dots\phi(x_n)\rangle. \tag{26}$$

Fixing some spacetime volume V, set $\epsilon = \chi_V$, where $\chi_V(x) = 1$ for $x \in V$ and is 0 otherwise. This gives

$$\langle Q(\Sigma)\phi(x_1)\dots\phi(x_n)\rangle = \sum_{x_i\in V}\langle\phi(x_1)\dots\delta\phi(x_i)\dots\phi(x_n)\rangle, \tag{27}$$

where $\Sigma = \partial V$ and $Q(\Sigma) = \int_\Sigma j$ as before. The physical meaning of Eq. (27) is clear: the charge operator $Q(\Sigma)$ defined on the surface Σ acts on the local operators inside the surface Σ as the symmetry generator and does nothing to local operators that are not enclosed by Σ. If we use only one field operator $\phi(x)$, we have

$$\langle Q(\Sigma)\phi(x)\rangle = L(\Sigma, x)\langle\delta\phi(x)\rangle, \tag{28}$$

where the linking number $L(\Sigma, x)$ counts whether x is enclosed by Σ. It is important to note that this expression is topological in the sense that it is insensitive to small deformations of the surface Σ as long as it does not cross the charged operator. This is the general feature of the action of symmetries on charged operators that generalizes to higher symmetries, as we will see.

For a finite symmetry operation, we have

$$\langle U(g,\Sigma)\phi(x)\rangle = R(g)\langle\phi(x)\rangle, \qquad L(\Sigma, x) = 1. \tag{29}$$

For $L(\Sigma, x) = 0$, it is obvious that $\phi(x)$ is left unchanged. If the group is a continuous Lie group such that $g = e^{tX}$, then $U(g,\Sigma) = e^{tQ(\Sigma)}$. However, the product of symmetry operators

is somewhat subtle to define here because of the auto-correlations of the operator Q. The appropriate definition of the product $O_1(\Sigma)O_2(\Sigma)\phi$ is given by deforming the second surface slightly, so that we have $O_1(\Sigma)O_2(\Sigma')\phi$, and then taking the limit where Σ' approaches Σ from the inside. It is clear that the regularized product is noncommutative and satisfies associativity, so we have $U(g,\Sigma)U(g',\Sigma) = U(gg',\Sigma)$. The noncommutativity is expected since we have non-Abelian symmetry groups.

Although the product is well defined, it is rather difficult to compute directly from $Q(\Sigma)$. In addition, we may have discrete symmetries for which there are no infinitesimal generators. It would be much easier if we could use finite transformations as a starting point. Indeed, define the transformation $\tilde{\phi}(x) = g_V \cdot \phi(x)$, where $g_V = g$ for $x \in V$ and is the identity otherwise. Assuming the invariance of the measure $D\phi$ under the transformation, we have

$$\langle F[g_V \cdot \phi]\rangle = \langle U(g,\Sigma)F[\phi]\rangle, \tag{30}$$

where $U(g,\Sigma) = \exp\big(-S[g_V^{-1}\cdot\phi] + S[\phi]\big)$. It is not hard to prove that $U(g,\Sigma)$ satisfies the associativity rule $U(g,\Sigma)U(h,\Sigma) = U(gh,\Sigma)$. In the case of a continuous symmetry, taking $g \to e$ recovers the original charge operator $Q(\Sigma)$.

GENERALIZED P-FORM GLOBAL SYMMETRY

Definitions of p-form global symmetry

Now we are ready to define a generalized global symmetry. In the case of an ordinary global symmetry, we have symmetry operators $U(g,\Sigma)$ defined on codimension-one surfaces Σ, and they act on charged operators of dimension 0, i.e. points. More generally, a generalized p-form global symmetry has symmetry operators $U(g,\Sigma)$ defined on a submanifold Σ with codimension $k+1$. The corresponding charged operators $O(C)$ are also extended objects, defined on a submanifold C with dimension k. Since $\dim C + \dim \Sigma = \dim M - 1$, we may also define the intersection number $L(\Sigma, C)$ for any pair of such manifolds C and Σ. In the special case where $k = 0$, the manifolds C are simply points, and we recover ordinary global symmetries. The transformation law reads

$$\langle U(g,\Sigma)O(C)\rangle = \begin{cases} R(g)\langle O(C)\rangle, & L(\Sigma,C) = 1 \\ \langle O(C)\rangle, & L(\Sigma,C) = 0 \end{cases} \tag{31}$$

We know that for an ordinary symmetry, the charged operators satisfy $\langle\phi(x)\rangle = R(g)\phi(x)$. The same is true for operators charged under generalized symmetry. To see this in a unified picture, assume that the spacetime manifold is periodic to avoid dealing with boundary conditions. In this case, as we mentioned when defining the linking number L, there are inequivalent ways of closing the surface Σ when it encloses a single charged operator on C. Either we close it by crossing the manifold C, for which we get $L = 1$, or we close it by pushing it to the periodic boundary of the torus and closing it there, for which we get $L = 0$. Since the answer must be the same regardless of the way we close Σ, we obtain

$$\langle O(C)\rangle = R(g)\langle O(C)\rangle, \tag{32}$$

which tells us that $\langle O(C)\rangle = 0$ if the charged operator transforms nontrivially under the symmetry action. This is the generalization of the familiar result of charge conservation for ordinary symmetries, which appears as a special case for $\dim\Sigma = 0$ here.

However, there is one important difference between higher-form symmetries and 0-form symmetries: higher-form symmetries are all Abelian. To see this, recall that for ordinary global symmetries, we define the regularized product of symmetry operators $U(g,\Sigma)$ and $U(h,\Sigma)$ on the codimension-one surface Σ by deforming the second operator to $U(h,\Sigma')$, where Σ' approaches Σ from the inside. The discontinuities of the deformation across the surface Σ give rise to non-Abelian group multiplication of the operators, i.e. $U(g,\Sigma)U(g',\Sigma) \neq U(g',\Sigma)U(g,\Sigma)$. The construction still works for higher symmetries: we can still obtain the regularized product by deforming the hypersurface Σ to Σ'. However, since the codimension of Σ is lower, there is no longer a notion of the "inside" and "outside" of the surface. As a result, the regularized product is Abelian.

The mathematical definition of generalized symmetry is clear, but it does not tell us how to search for such symmetries in a physical theory. To do that, we again rely on analogies with ordinary symmetries. Let us focus on the continuous case, where the generator form of Eq. (31) reads

$$Q(\Sigma)O(C) = L(\Sigma,C)\delta O(C). \tag{33}$$

In the case of an ordinary symmetry, the above equation comes from the Schwinger-Dyson equation (25), obtained by transforming the local degrees of freedom in $S[\phi]$ according to the

symmetry action. The current j is defined so that, since differential forms allow us to manipulate tensors without dealing with indices, we will mostly use the current form j instead of the current J, which is related to j by the Hodge dual $j = \star J$:

$$\delta S = -\int d\epsilon \wedge j = \int \epsilon \, dj. \tag{34}$$

The classical equations of motion force $dj = 0$ up to quantum contact terms with insertions of local charged operators, which gives us the infinitesimal transformation rule:

$$dj \, O = \delta O. \tag{35}$$

In the case of a generalized symmetry, the direct analog of Eq. (34) is

$$\delta S = \int \epsilon \wedge dj. \tag{36}$$

Since the local degrees of freedom are point-like operators, the symmetry might not act on them directly. As a result, it may be difficult to give a meaning to δS in that case. However, as long as we can find local current forms j that satisfy $dj = 0$, we know that they can be candidates for higher-symmetry generators with appropriate charged operators. We could still formally write $\delta S = \int \epsilon \wedge dj$, and the contact relation is still given by the Schwinger-Dyson equations.

Note that instead of local functions, ϵ is now a local differential form. This is required since we want to integrate ϵ on some extended charged objects instead of points. The generalized symmetry associated with a current form of degree k has form degree $q = n - k - 1$, so we have $\deg \epsilon = q$.

Examples of Generalized Symmetry

4D Free Maxwell Theory

Let us start with the action of the 4D free Maxwell theory in the absence of electric charge:

$$S = \frac{1}{2e^2} \int F \wedge \star F \tag{37}$$

There are two natural 2-form currents in the theory:

$$j_m = \frac{F}{2\pi}, \qquad j_e = \frac{\star F}{e^2} \tag{38}$$

The current conservation laws are guaranteed by the Bianchi identity and the equations of motion of the A field.

Let's start by considering the "magnetic current" j_m. The symmetry charge $Q(\Sigma)$ is defined with respect to a 2-dimensional surface Σ:

$$Q_m(\Sigma) = \frac{1}{2\pi}\int_\Sigma F, \qquad U_m(\Sigma,\alpha) = \exp\left(\frac{i\alpha}{2\pi}\int_\Sigma F\right). \tag{39}$$

Since $F = dA$, it seems naively that the symmetry is trivial, since we get 0 when integrate over closed surfaces. This is not true when the background space has non-trivial topological structures, in which case the connection A is only defined locally. As a result, the charged operators will be topological defects of the background manifold. In fact, from the theory of $U(1)$ bundles, we know that $Q(\Sigma)$ is an integer that counts the number of magnetic monopoles enclosed by the surface Σ, which are dimension-one line defects of the $U(1)$ bundle. The charged operators are therefore the worldlines of the magnetic monopoles. For a surface Σ that encloses m monopoles, we have $Q = n$, $U(\Sigma,\alpha) = e^{in\alpha}$.

The story is the same for the electric charge operator j_e, since in 4D the electric field and magnetic field are dual. The charge

$$Q_e(\Sigma) = \frac{1}{2\pi}\int_\Sigma \star F, \qquad U_e(\Sigma,\alpha) = \exp\left(\frac{i\alpha}{2\pi}\int_\Sigma \star F\right). \tag{40}$$

counts the worldlines of electrons enclosed by the surface. The charged operators are Wilson loops [6]

$$W_e(C) = e^{i\int_C A} \tag{41}$$

of the gauge field A. To derive the transformation rule, we can use a shift of variables $\tilde{A} = A + \alpha$ for a local one-form field α. In this case, we can derive the variation of the action directly:

$$\delta S = \delta\left(\frac{1}{2e^2}\int F\wedge\star F\right) = \frac{1}{e^2}\int \alpha\wedge(d\star F). \tag{42}$$

In the presence of the Wilson line operator $W_e(C)$, we have

$$\int_M \alpha\wedge dj_e = i\int_C \alpha = i\int_M \alpha\wedge\theta_C, \tag{43}$$

where θ_C is the form dual to the line C. Since α is arbitrary, we have

$$dj_e W_e(C) = i\theta_C W_e(C), \quad Q_e(\Sigma) = iL(\Sigma,C)W_e(C). \tag{44}$$

The resulting transformation rule of $W_e(C)$ is

$$U_e(\Sigma,\alpha)W_e(C) = e^{in\alpha}W_e(C), \tag{45}$$

where $n = L(\Sigma,C)$ is the total charge. This is exactly what we expect from a $U(1)$ global symmetry.

Z2 Lattice Gauge Theory in 3D

In this section, we discuss an example of generalized symmetry in a lattice theory: the 3D Z2 lattice gauge theory. In a 3D lattice, the degrees of freedom of the theory are numbers $s_l = \pm 1$ with $s = 1/2$ living on the links of the lattice. It is a classical statistical model with Hamiltonian

$$H = \sum_{p \in P} K_p B_p, \quad B_p = \prod_{l \in p} s_l, \tag{46}$$

where P is the set of plaquettes of the lattice, and K_p are the couplings on the plaquettes p. The theory has a local Z_2 gauge symmetry, generated by a function $\alpha_v = \pm 1$ on the vertices. The gauge transformation takes $s_{ij} \to \alpha_i \alpha_j s_{ij}$, where i, j is a pair of adjacent vertices. For every closed loop C, we have the Wilson line $W_e(C)$, defined as

$$W_e(C) = \prod_{l \in C} s_l. \tag{47}$$

It is clear that $W_e(C)$ is gauge invariant because the factor α_i on every vertex contributes an even number of times. In particular, since plaquettes are special loops, the Hamiltonian is gauge invariant.

Now we introduce one-form symmetry operators $W_m(\Gamma)$ that are also line operators. Here Γ is required to be a loop on the dual lattice, i.e. the lattice formed by the plaquettes of the original lattice. $W_m(\Gamma)$ is defined as a defect that flips the coupling $K_p \to -K_p$ on every plaquette it crosses. It is immediately clear that $W_m(\Gamma)$ is topological, since for a plaquette γ of the dual lattice, the action of $W_m(\gamma)$ flips all the plaquette couplings K_p connected to the link l that pierces γ. By taking a shift of variables $s_l \to -s_l$, we can undo the effect of $W_m(\gamma)$, which means it is trivial. As a result, we are allowed to deform the loop defect $W_m(\Gamma)$ across space by multiplying it with $W_m(\gamma)$ on dual plaquettes, which makes it topological. The nontrivial effect occurs when there is already an existing Wilson line $W_e(C)$ in the system. In this case, the variable shift $s_l \to -s_l$ sends $W_e(C)$ to $-W_e(C)$ when C intersects the dual plaquette γ. As a result, we obtain the desired contact relation between W_m and W_e:

$$W_m(\Gamma) W_e(C) = (-1)^{L(\Gamma, C)} W_e(C) \tag{48}$$

This tells us that W_e is the Z_2 charged line operator that transforms under the action of $W_m(\Gamma)$, which is a generalized one-form symmetry.

Spontaneous Breaking of Generalized Symmetry

Since we have defined generalized symmetry by analogy with ordinary 0-form symmetry, it should not be surprising that generalized symmetry can also be spontaneously broken. In this section, we shall explore the consequences of this possibility.

In the case of an ordinary symmetry, **Spontaneous Symmetry Breaking**(SSB) occurs when we have a degenerate space of ground states that transforms nontrivially under the symmetry group G. For a single ground state, a local charged operator $O(x)$ can acquire a nonzero expectation value $\langle O(x)\rangle \neq 0$. However, when we evaluate the expectation value in the partition function, since all degenerate ground states contribute with equal weight, the resulting expectation value is averaged over all ground states and is still formally symmetry preserving, which gives $\langle O(x)\rangle = 0$. To distinguish ground states with and without SSB, we often consider the two-point functions $\langle O(x)O(y)\rangle$ of local charged operators. Since for a single ground state we expect no long-range entanglement, the correlators should decay exponentially:

$$\langle O(x)O(y)\rangle_\psi \sim \langle O(x)\rangle_\psi \langle O(y)\rangle_\psi, \quad |x-y| \to \infty \tag{49}$$

Therefore, if the correlator evaluated by the partition function satisfies

$$\langle O(x)O(y)\rangle \sim O(1), \quad |x-y| \to \infty, \tag{50}$$

we know that there must be long-range entanglement due to degenerate ground states, and that we are in a phase with broken symmetry.

The same logic applies to generalized symmetry. For simplicity, we focus on the case where the charged operators are line operators. Consider the Wilson lines $W(C)$ discussed in previous section. It turns out that there are two possible asymptotic behaviors of $\langle W(C)\rangle$ when the size of the loop C goes to infinity. We either have $-\log W(C) \sim \text{Area}(C)$ or $-\log W(C) \sim \text{Perimeter}(C)$. These behaviors are called the "area law" and the "perimeter law", respectively. To interpret the result, work with periodic time $\tau \in [0, \beta)$, and consider the special spacetime loop C consisting of a path C_1 running from $\tau = 0$ to $\tau = \beta$ at $x = 0$ and another path running from $\tau = \beta$ to $\tau = 0$ at $x = x_0$. The full path is $C = C_1 + C_2$.

Since C_1, C_2 are charged under the symmetries, periodic boundary conditions in space guarantee that

$$\langle W(C_{1,2})\rangle = 0 \tag{51}$$

Therefore, $\langle W(C)\rangle = \langle W(C_1)W(C_2)\rangle$ probes the long-range entanglement in the system. In the area-law case, we have

$$\langle W(C)\rangle \sim e^{-\beta|x|}, \tag{52}$$

which gives the desired exponential decay. The symmetry is therefore unbroken. This is sometimes called the "confined" phase. If we imagine the Wilson lines as worldlines of particles charged under the Z_2 gauge symmetry that are separated by a distance x, their coupling to the gauge fields is exactly the Wilson line. As a result, we have $\beta|x| = \beta\,\delta E$. Here δE is the binding energy of the particle configuration in the Born-Oppenheimer approximation. We can see that in the area-law case, the energy of the charged-particle configurations diverges linearly as we increase the distance between the particles. The charged particles therefore never appear as excitations in the IR theory. That's what "confinement" means.

On the other hand, when the perimeter law is true, then

$$W(C) \sim e^{-\beta}. \tag{53}$$

As a result, the correlator is roughly independent of the spatial distance x as $x \to \infty$. (In fact, it has a power law decay if one works out the details) This shows that there is long-range entanglement in the system, and the generalized symmetry is spontaneously broken. In fact, this is exactly the situation described by topological order: there are degenerate ground states in the system that are related to one another only under the action of extended charged operators.

Photon As a Goldstone Boson

Goldstone's theorem [7] tells us that when a continuous ordinary global symmetry is broken, there is always a massless Goldstone boson. On the other hand, we also know another important massless boson in nature: the photon. In the usual logic of quantum field theory, the masslessness of photons is protected by gauge symmetries, instead of a broken global symmetry. Generalized symmetry, however, provides a natural way to view the photon as the Goldstone boson associated with the breaking of generalized one-form global symmetries. To see this, let us first review how we derive the Goldstone boson from ordinary spontaneous symmetry breaking. For a system with an ordinary $U(1)$ symmetry, we can make the symmetry local by coupling it to some background $U(1)$ gauge field $\mathcal{A}$, with the coupling term

$$\delta S \sim \int j \wedge \mathcal{A}. \tag{54}$$

When spontaneous symmetry breaking occurs, the effective action acquires a mass term $\sim \mathcal{A}^2$. However, due to gauge invariance, this term must be dressed by some scalar field ϕ, so that the combined gauge-invariant mass term is

$$\delta\mathcal{L} \sim (\mathcal{A} + d\phi)^2. \tag{55}$$

Setting the background field $\mathcal{A} = 0$ gives us the kinetic term of the Goldstone boson ϕ.

For a system with q form generalized symmetries, we can also couple it to background gauge fields $\mathcal{B}$, so that

$$\delta S \sim \int j \wedge \mathcal{B}. \tag{56}$$

Since $\deg j = n - q - 1$, we have $\deg B = q + 1$, so the $\mathcal{B}$ fields are $q + 1$-form gauge fields. Spontaneous symmetry breaking gives the effective mass term

$$S_{eff} \sim \int (\mathcal{B} + da)^2, \tag{57}$$

where a is the analog of the ϕ field, which is a differential q form that is necessary to make the expression gauge invariant. The gauge transformation takes $a \to a + \alpha$, $\mathcal{B} \to \mathcal{B} - d\alpha$. Setting $\mathcal{B} = 0$ gives us the leading effective action of the a fields:

$$S_{eff} \sim \int da \wedge \star da \tag{58}$$

For $k = 1$, this indeed gives us the familiar Maxwell action of the photon. For higher generalized symmetry, this implies higher (q-form) abelian gauge fields with Maxwell action (58).

SUMMARY

In this paper, we give an introductory discussion of generalized global symmetries in quantum field theory. We first review the mathematical background needed for the later discussion, including differential forms, Poincaré dual forms, and linking numbers. These tools allow us to describe symmetry operators and charged operators in a geometric way. We then revisit ordinary 0-form global symmetries, starting from Noether's theorem in classical field theory and continuing to the Schwinger-Dyson equations in quantum field theory.

After this preparation, we introduce higher-form global symmetries as a natural generalization of ordinary global symmetries, for which symmetry operators are supported on higher-dimensional submanifolds, and charged operators are extended objects rather than local point-like operators. We explain how the action of a higher-form symmetry is determined by the

linking number between the symmetry operator and the charged operator. We then discuss several examples, including the electric and magnetic one-form symmetries of four-dimensional free Maxwell theory and the one-form symmetry in three-dimensional Z_2 lattice gauge theory. Finally, we describe spontaneous symmetry breaking for generalized symmetries and also explained how the photon can arise as a Goldstone boson associated with a broken one-form global symmetry.

The discussion in this paper focuses on ordinary higher-form global symmetries. There are many further generalizations of symmetry that are not covered here, including higher-group symmetries, categorical symmetries, and non-invertible symmetries [8–11]. These structures extend the idea of generalized symmetry beyond the cases considered in this paper and play an important role in more modern developments.

AI STATEMENT

Chatgpt helped me finding references for the subject and explaining ideas including the construction/action of the symmetry operators, defects, etc. It also helped me in the writing the final paper. I wrote the first draft and then asked it to generate a polished version by "correcting the grammatical errors/inappropriate expressions, but do not change the formulas/citations." I checked the result and did the final revision myself.

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