Conformal Field Theory and the Bootstrap

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Physics 332 — Quantum Field Theory III

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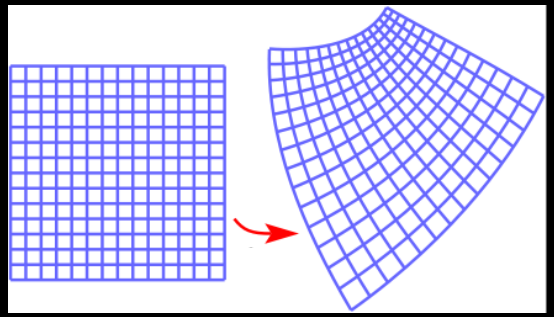
Motivation: RG fixed points → scale invariance

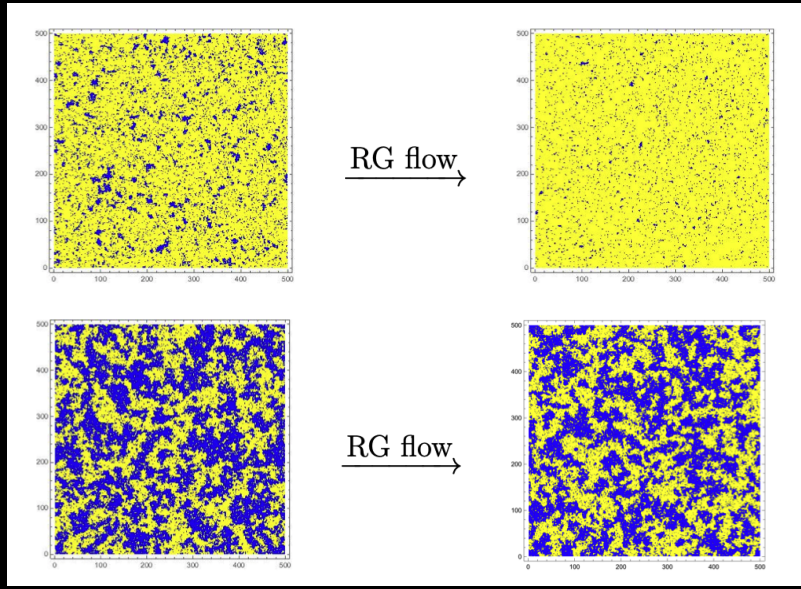
Endpoints of RG flow are scale-invariant fixed points.

We expect (under weak assumptions) such scale-invariant theories are always **Conformal Field Theories (CFTs):**

A CFT is a quantum or statistical field theory that is invariant under local *rescalings, or conformal transformations.* (Cares about angles, not distances)

Examples: critical Ising model, phi-4 at Wilson-Fischer fixed point





RG flow on the Ising model

The full conformal group (d ≥ 3)

Transformation	Generator	Finite form
Translation	P_μ	$x^\mu \to x^\mu + a^\mu$
Rotation	$M_{\mu\nu}$	$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$
Dilatation	D	$x^\mu \to \lambda \, x^\mu$
Special conformal	K_μ	$x^\mu \to (x^\mu + b^\mu x^2) / (1 + 2\, b\cdot x + b^2 x^2)$

Generators $\{ P_\mu, M_{\mu\nu}, D, K_\mu \}$ span the conformal algebra $so(d+1, 1)$.

Conformal symmetry fixes the 2-point function

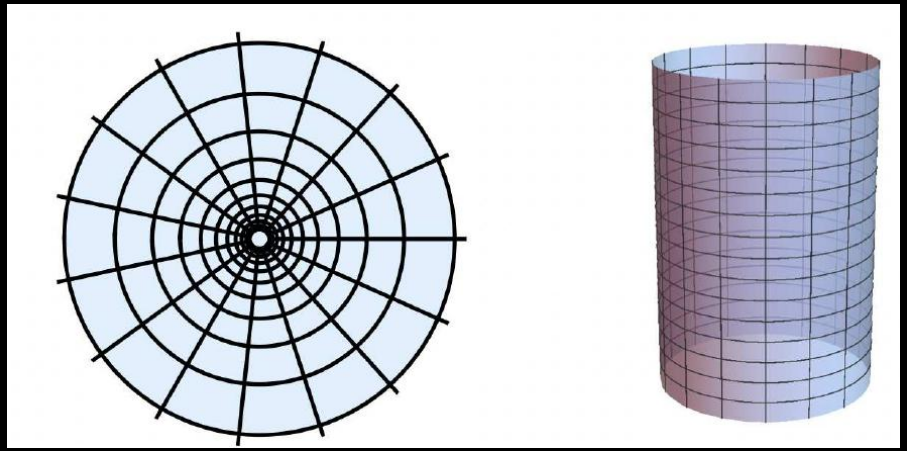
For scalar primaries $\mathcal{O}_i$, $\mathcal{O}_j$ of dimensions Δ_i, Δ_j, the two-point function is completely determined:

$$\langle \mathcal{O}_i(x)\, \mathcal{O}_j(0) \rangle = \frac{\delta_{ij}}{|x|^{2\Delta_i}} .$$

- **translation + rotation invariance** ⇒ depends only on $|x - y|$;
- **dilatation invariance** with weights Δ_i, Δ_j ⇒ power law $|x|^{-(\Delta_i + \Delta_j)}$;
- **special conformal invariance** ⇒ forces $\Delta_i = \Delta_j$;
- **operator normalization** ⇒ the coefficient is δ_{ij}.

Radial quantization

- We foliate our theory with time evolving **radially**
- Hamiltonian becomes the dilatation operator *D*
- Eigenstates of *D* have definite *scaling dimension* Δ : $D\,|\Delta\rangle = \Delta\,|\Delta\rangle$
- Lowest-weight eigenstates are called *primaries*, annihilated by K_μ .



Radially evolving CFT ⟺ **CFT on a cylinder**

The state–operator correspondence

Insert a local operator at the origin; the path integral over its neighborhood gives a state on the surrounding sphere:

$$\mathcal{O}(0) \longleftrightarrow |\mathcal{O}\rangle$$

Every local operator at the origin defines a state on the unit sphere,
and conversely every state on the sphere is created by a local operator.

This lets us expand (products of) local operators in a basis!

The Operator Product Expansion

Inside a ball around the origin, a product of two operators decomposes as a sum over other operators:

$$\mathcal{O}_i(x)\,\mathcal{O}_j(0) \;=\; \sum_k f_{ijk}\, C_{ijk}(x,\partial)\,\mathcal{O}_k(0)$$

- sum runs over conformal *primaries* $\mathcal{O}_k$ dual to primary states
- C_{ijk} packages descendants and is fixed by conformal symmetry once Δ_i, Δ_j, Δ_k and d are specified
- the only dynamical data are $\{\Delta_k, f_{ijk}\}$ — the **CFT data**
- Fact I won't prove: in an orthonormal basis, f_{ijk} are real, and $f_{iik}^2 \geq 0$.

Solving the 4-point function with the OPE

Apply the OPE to pairs (1, 2) and (3, 4) of identical scalars φ :

$$\left\langle \overbrace{\phi(x_1)\,\phi(x_2)}\ \overbrace{\phi(x_3)\,\phi(x_4)} \right\rangle$$

$$\langle \mathcal{O}_i(x)\,\mathcal{O}_j(0)\rangle \;=\; \frac{\delta_{ij}}{|x|^{2\Delta_i}}\,.$$

Insert the OPE on both pairs:

$$= \sum_{k,k'} f_{\phi\phi k}\, f_{\phi\phi k'}\, C_{\phi\phi k}(x_{12},\partial_2)\, C_{\phi\phi k'}(x_{34},\partial_4)\, \langle \mathcal{O}_k(x_2)\,\mathcal{O}_{k'}(x_4)\rangle$$

The two-point function $\langle\, \mathcal{O}_k\, \mathcal{O}_{k'} \,\rangle \;\propto\; \delta_{kk'}$ *forces* $k = k'$*, collapsing the double sum:*

$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle \;=\; \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}}\; \mathcal{G}^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(\vec{x})$$

$g^{(s)}_{\Delta_\phi,\,\Delta,\,\ell}(\vec{x})$ = ***conformal block*** *: known function fixed by conformal symmetry.*

OPE associativity $\Longrightarrow$ The bootstrap

A four-point function admits multiple OPE decompositions; they must all agree.

$$\underbrace{\left\langle \overbrace{\phi(x_1)\,\phi(x_2)}\;\overbrace{\phi(x_3)\,\phi(x_4)}\right\rangle}_{s\text{-channel}\;\;(1,2)\,(3,4)} \;=\; \underbrace{\left\langle \overbrace{\phi(x_1)\;\overbrace{\phi(x_2)\,\phi(x_3)}\;\phi(x_4)}\right\rangle}_{t\text{-channel}\;\;(1,4)\,(2,3)}$$

implies:

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} \left[\mathcal{G}^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(x_1,x_2,x_3,x_4) \;-\; \mathcal{G}^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(x_1,x_4,x_3,x_2)\right] \;=\; 0$$

This equality — the **crossing equation** — is an infinite set of constraints on the CFT data {Δ, f}.

The conformal bootstrap aims to determine what CFT data is consistent with this.

How can we identify what CFT data solves the crossing equation?

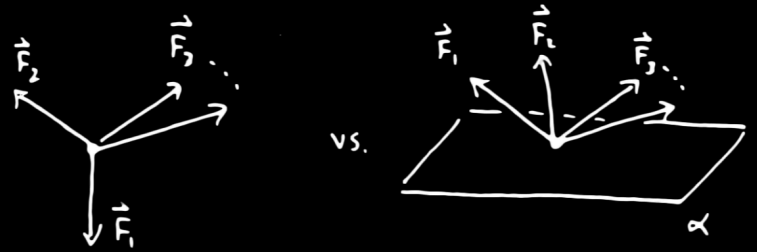
- We have a set of known functions

$$F^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(\vec{x}) \equiv \mathcal{G}^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(x_1,x_2,x_3,x_4) - \mathcal{G}^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(x_1,x_4,x_3,x_2)$$

- Think of these as vectors in an (infinite dimensional) vector space, subject to the following constraint:

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}}\, F^{\Delta_\phi}_{\Delta,\ell}(\vec{x}) = 0\,, \qquad f^2_{\phi\phi\mathcal{O}} \geq 0$$

- Vectors must be able to cancel out!



Procedure and example for identical scalars

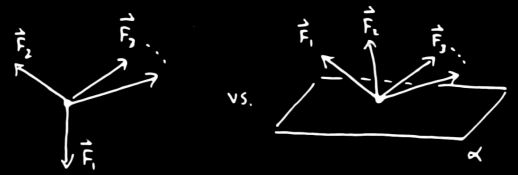
Choose constraints and a question: Take a 2d CFT with a real scalar primary ϕ of dimension $\Delta_\phi = 0.20$. Which weights $\{\Delta, \ell\}$ appear in the $\phi \times \phi$ OPE?

Project to a subspace. To make the procedure tractable, map the infinite-dim crossing equation onto a 2d subspace via (for some x1, x2):

$$\vec{v}(F^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}) \in \mathbb{R}^2, \quad v_1 = F^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(\vec{x}_1), \quad v_2 = F^{\Delta_\phi}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(\vec{x}_2)$$

Find a separating plane. Plot the curves $\vec{v}(F_{\Delta,\ell})$ for all ℓ , Δ and search for a line through the origin such that all vectors lie on one side. (Can be done by applying a linear functional)

Constrain the data. CFT data must lead to vectors on both sides of any separating plane.

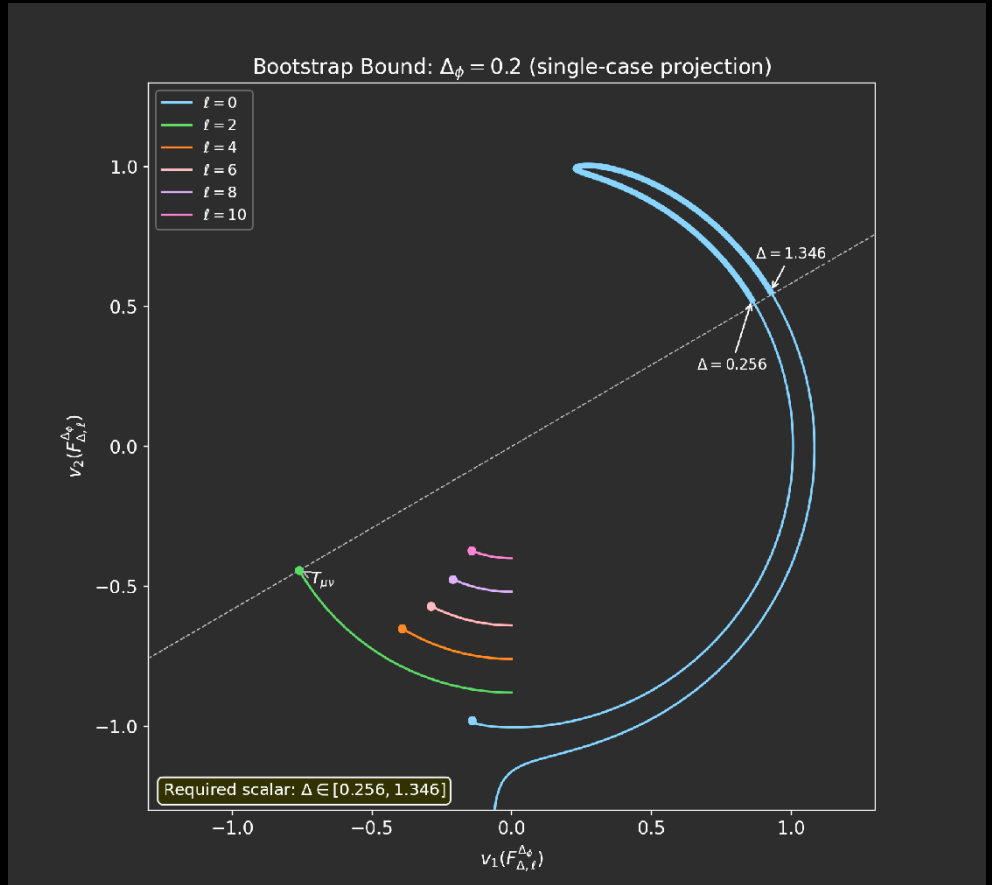


Example for identical scalars: Plot

- **Each point is a possible operator in the ϕ × ϕ OPE**:
  - colors give ℓ
  - line parameter is Δ
- Must have points on either side of the separating plane
- Therefore, in a unitary 2d CFT with real ϕ of dimension 0.2, the ϕ × ϕ OPE must contain a dimension

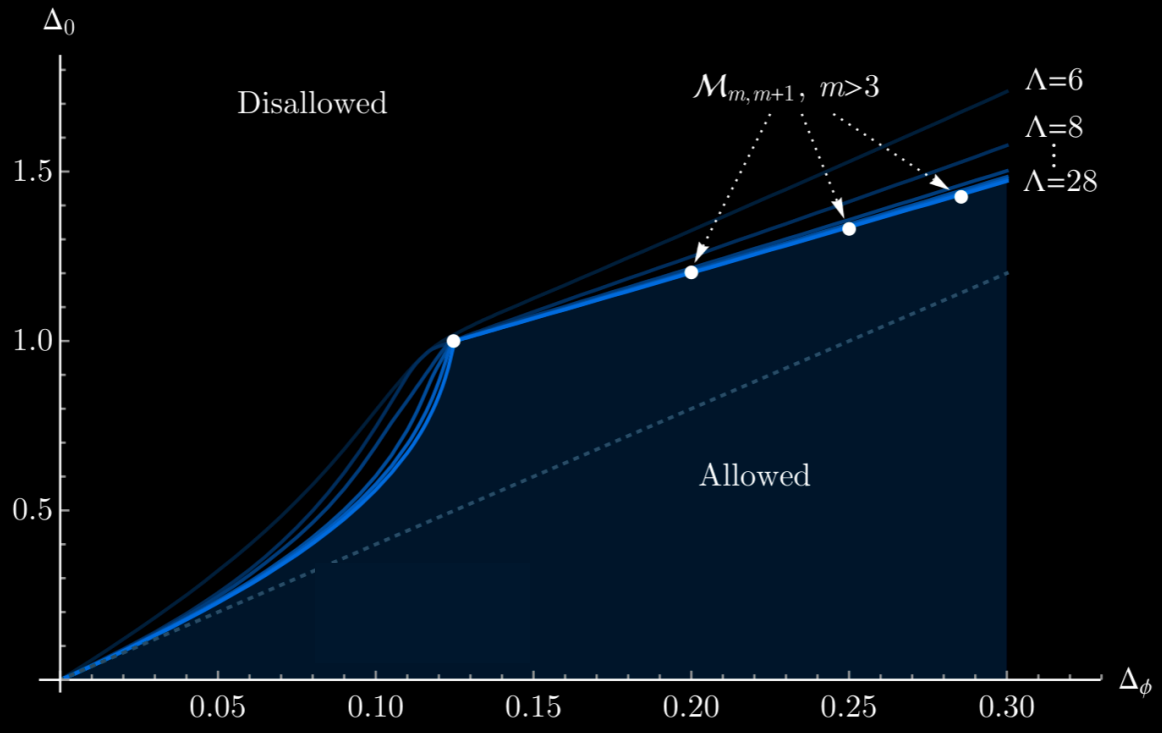
$$\Delta \in [0.2564,\ 1.3463]$$

- Not bad for an arbitrary choice of constraints running on my laptop



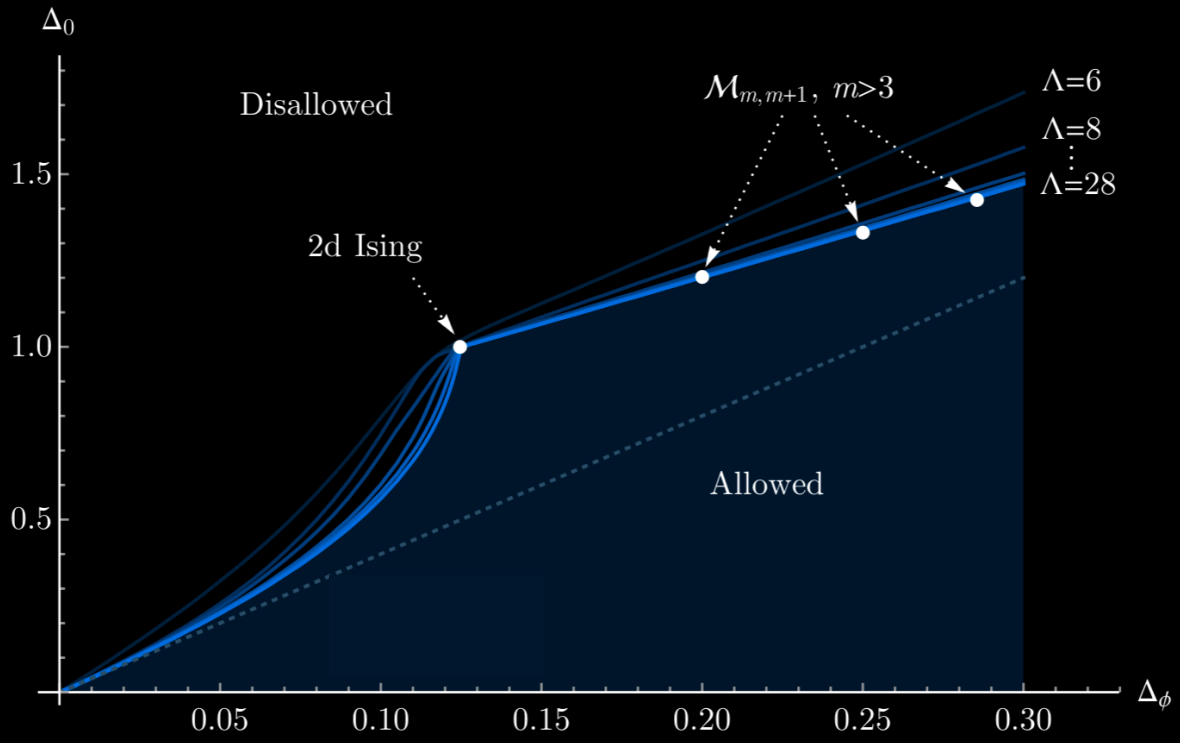
Plotting across all possible Δ_ϕ

- Increase subspace dimension and repeat this procedure with a ton of compute
- What are the best bounds we can obtain for different Δ_ϕ?
- A kink appears in the bound…



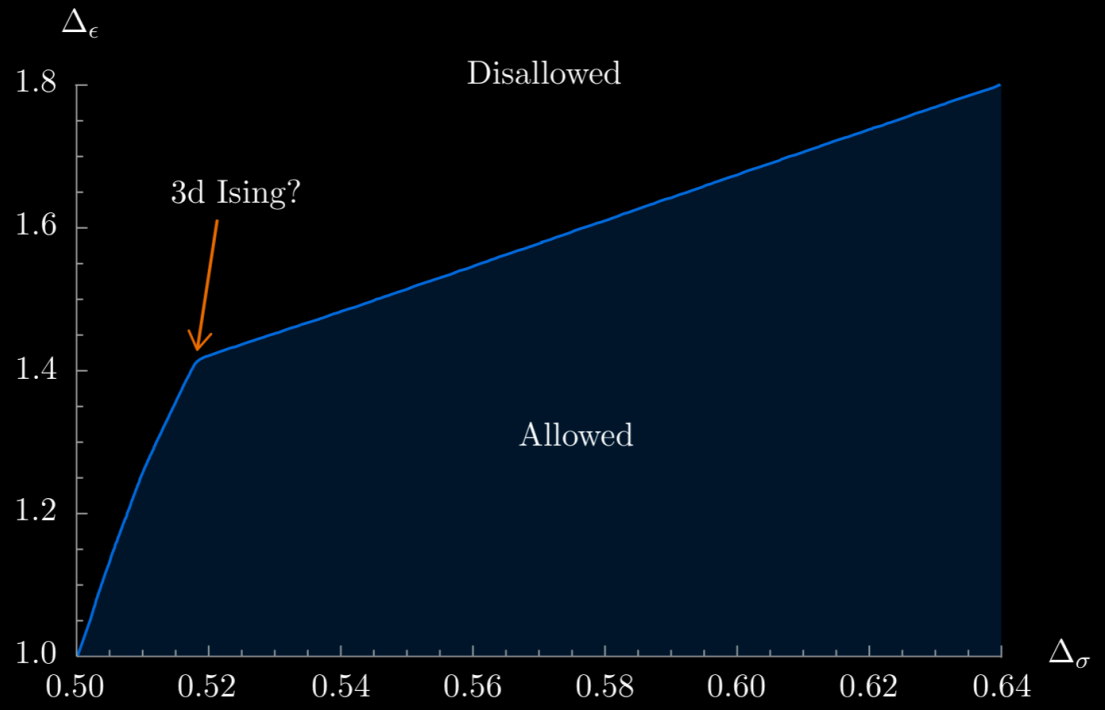
The Ising Model!

- 2d Ising and other solved models seem to exactly saturate the bootstrap bound
- Interesting, but 2d Ising is solved…



The 3D Ising kink

- Same thing seems to work in 3d, but now we don't know the critical exponents analytically
- Can we constrain the position of the kink?



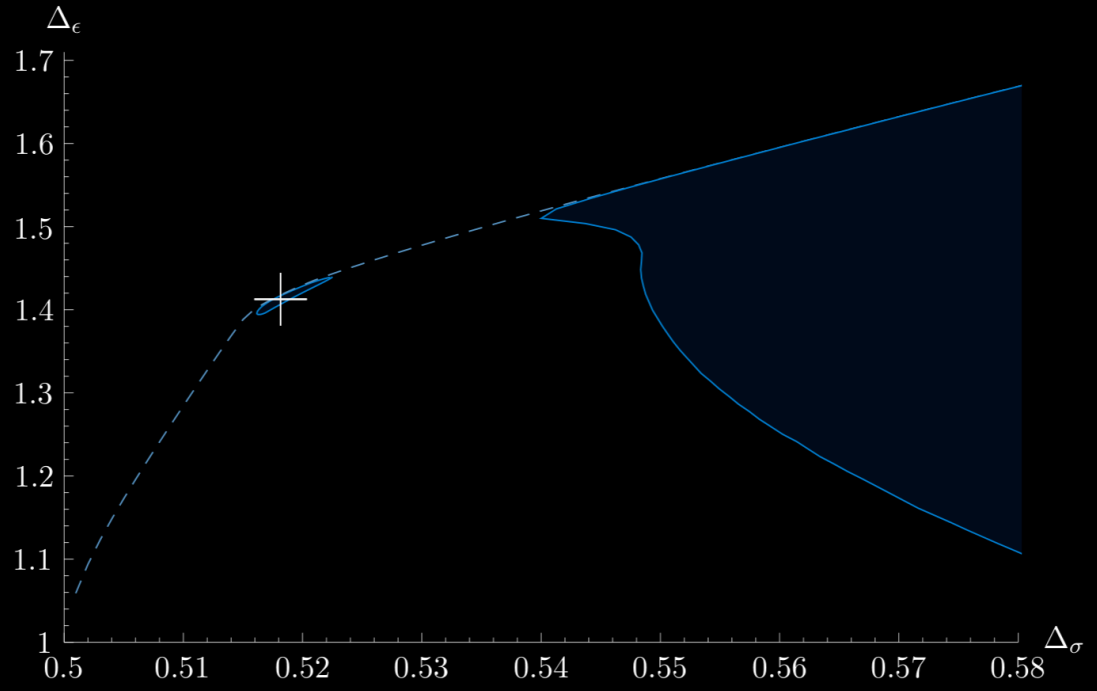
The 3D Ising island

- There are only two relevant scalar fields in the Ising CFT: the spin operator σ and the energy operator ϵ
- Applying the bootstrap to other 4 point functions lets us constrain the space even further:

$$\langle \sigma(x_1)\,\sigma(x_2)\,\sigma(x_3)\,\sigma(x_4)\rangle$$
$$\langle \sigma(x_1)\,\sigma(x_2)\,\epsilon(x_3)\,\epsilon(x_4)\rangle$$
$$\langle \epsilon(x_1)\,\epsilon(x_2)\,\epsilon(x_3)\,\epsilon(x_4)\rangle$$

- Only an island of validity: **universality**

Sharpest known constraint on 3d Ising critical exponents! 29 s.f.



Future directions

- Does the bootstrap island for 3d ising converge to a point?
- Analytic bootstrap, large-spin expansions and limits. Complementary to the numerical bootstrap
- Extending the bootstrap to gauge theories, non-lagrangian CFTs
- AdS-CFT applications: ensemble of theories dual to 3d gravity
- Better computational methods