# An Overview of Montonen-Olive Duality in $\mathcal{N} = 4$ Super Yang-Mills Theory

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# Classical Computations

- Dirac charge quantization;
- BPS bound for ‘t Hooft Monopole and Julia-Zee Dyon;
- Problems in transferring to a QFT.

# Maxwell Symmetry

$$\nabla \cdot \vec{\mathrm{E}} = 0 \qquad \vec{\mathrm{E}} \rightarrow \vec{B}, \vec{B} \rightarrow -\vec{\mathrm{E}} \qquad \nabla \cdot \vec{B} = 0$$

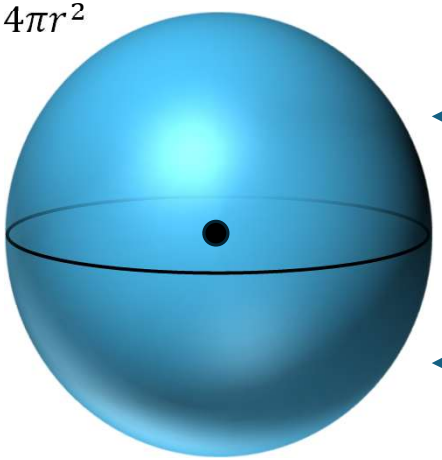
$$\nabla \times \vec{B} - \frac{\partial \vec{\mathrm{E}}}{\partial t} = 0 \qquad \nabla \times \vec{\mathrm{E}} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{\mathrm{E}} = \rho_e \qquad ? \qquad \nabla \cdot \vec{B} = \rho_{\mathrm{m}}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{\mathrm{E}}}{\partial t} = \vec{\mathrm{J}}_e \qquad \nabla \times \vec{\mathrm{E}} + \frac{\partial \vec{B}}{\partial t} = \vec{\mathrm{J}}_{\mathrm{m}}$$

# Dirac Charge Quantization

$$\vec{B} = \frac{g\hat{r}}{4\pi r^2}$$



$$\vec{A}_N = \frac{g}{4\pi r^2}\frac{(1-cos(\theta))}{sin(\theta)}\widehat{e_\phi}$$

$$\vec{A}_S = -\frac{g}{4\pi r^2}\frac{(1+cos(\theta))}{sin(\theta)}\widehat{e_\phi}$$

At Equator: $\vec{A}_N - \vec{A}_S = -\nabla\chi; \chi = -\frac{g}{2\pi}\phi$

$$g = \int_E (\vec{A}_N - \vec{A}_S)\cdot d\vec{l} = \chi(0) - \chi(2\pi)$$

$$\therefore e^{-ieg} = 1 \Rightarrow eg = 2\pi n$$

# BPS Bound – Monopole and Dyon

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\left(D_\mu\phi^a\right)(D^\mu\phi^a) - \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2$$

$$E = \int d^3x \left[\frac{1}{2}B^a_i B^a_i + \frac{1}{2}(D_i\phi^a)(D_i\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2\right], \qquad B^a_i = \frac{1}{2}\epsilon_{ijk}F^a_{jk}$$

$$E = \frac{1}{2}\int d^3x\,(B^a_i \mp D_i\phi^a)^2 \pm \int d^3x\, B^a_i D_i\phi^a \geq \left|\oint_{S^2_\infty} dS_i \phi^a B^a_i\right|, \qquad n = \frac{1}{8\pi}\int_{S^2_\infty} \epsilon_{ijk}\epsilon^{abc}\widehat{\phi^a}\,\partial_j\widehat{\phi^b}\partial_k\widehat{\phi^c}dS_i, \qquad \widehat{\phi^a} = \frac{\phi^a}{|\phi|}$$

$$E \geq \left|\oint_{S^2_\infty} dS_i \phi^a B^a_i\right| = \frac{1}{2g}\oint_{S^2_\infty} dS_i \epsilon_{ijk}\epsilon^{abc}\widehat{\phi^a}\partial_j\widehat{\phi^b}\partial_k\widehat{\phi^c}\,,$$

$$\therefore E \geq \frac{4\pi v}{g}|n|$$

Saturated when: $B^a_i = \pm D_i\phi^a$



$$E \geq \frac{4\pi v}{g}\sqrt{Q_m^2 + Q_e^2}$$

# Problem: QFT from Classical

Problem 1: Quantum Corrections

- Additional or modified terms
- Possible problems in the derivation

$$E \geq \frac{4\pi v}{g}\sqrt{Q_m^2 + Q_e^2}$$

$\beta(g)$

Problem 2: Spin Configuration

<u>Magnetic Piece</u>

Rotationally Invariant → Spin-0

<u>Electric Piece</u>

W-Boson → Spin-1

# Model: $\mathcal{N} = 4$ Super Yang-Mills Theory

- Generalizing the BPS bound
- Vanishing of the beta function
- Electric and magnetic spin structure

# Exactness of the Bound

$$\{Q^{\mathrm{I}}_{\alpha}, \overline{Q_{\beta J}}\} = 2(\sigma^{\mu})_{\alpha\beta}\, \mathrm{P}_{\mu}\, \delta_{\mathrm{J}}{}^{\mathrm{I}} \qquad \{Q^{\mathrm{I}}_{\alpha}, Q^{\mathrm{J}}_{\beta}\} = 2\epsilon_{\alpha\beta}\, \mathrm{Z}^{\mathrm{IJ}} \qquad [Q^{\mathrm{I}}_{\alpha}, \mathrm{P}_{\mu}] = [\overline{Q_{\beta J}}, \mathrm{P}_{\mu}] = [\mathrm{Z}^{\mathrm{IJ}}, \mathrm{X}] = 0$$

➡ Key Object: Central Charges $\mathrm{Z}^{\mathrm{IJ}}$

$$Z = \begin{bmatrix} 0 & q_1 & 0 & 0 & \dots & 0 & 0 \\ -q_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & q_2 & \dots & 0 & 0 \\ 0 & 0 & -q_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & q_{\mathcal{N}/2} \\ 0 & 0 & 0 & 0 & \dots & -q_{\mathcal{N}/2} & 0 \end{bmatrix}$$

Ladder Operators:

$$a_{\alpha} = \frac{1}{\sqrt{2}}\left(Q^{1}_{\alpha} + \epsilon_{\alpha\beta}\overline{Q^{2}_{\beta}}\right) \qquad b_{\alpha} = \frac{1}{\sqrt{2}}\left(Q^{1}_{\alpha} - \epsilon_{\alpha\beta}\overline{Q^{2}_{\beta}}\right)$$

$$a^{\dagger}_{\alpha} = \frac{1}{\sqrt{2}}\left(\overline{Q^{1}_{\alpha}} + \epsilon_{\alpha\beta}Q^{2}_{\beta}\right) \qquad b^{\dagger}_{\alpha} = \frac{1}{\sqrt{2}}\left(\overline{Q^{1}_{\alpha}} - \epsilon_{\alpha\beta}Q^{2}_{\beta}\right)$$

➡ $M \geq \frac{1}{2}|Z|$ (when restricting to one charge)

# Defining the Model

$$S_{10} = \int d^{10}x \left( -\frac{1}{4} F^a_{MN} F^{MN\,a} + \frac{i}{2} \overline{\lambda^a} \Gamma \cdot D \lambda^a \right)$$

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g \epsilon_{abc} A^b_M A^c_N$$

$$\lambda = C \overline{\lambda^T} \text{ and } \lambda = \pm \Gamma_{11} \lambda$$

$$S_4 = \int d^4x \left[ -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\,a} + \frac{1}{2} (D^\mu A^a_i)(D_\mu A^a_i) + \frac{1}{2} (D^\mu B^a_j)(D_\mu B^a_j) \right.$$
$$\left. + \frac{1}{2} i (\overline{\psi^a} \gamma^\mu D_\mu \psi^a) - \frac{g}{2} \overline{\psi^a} \epsilon^{abc} (i \alpha^i A^b_i + i \beta^j B^b_j \gamma_5) \psi^c - V(A,B) \right]$$

$$V(A,B) = -\frac{g^2}{4} ([A_i, A_j], [A_i, A_j]) - \frac{g^2}{4} ([B_i, B_j], [B_i, B_j]) - \frac{g^2}{2} ([A_i, B_j], [A_i, B_j])$$

$$A_{3+i} \equiv A_i, \qquad A_{6+j} \equiv B_j, \qquad \mathrm{i}, j = 1,2,3$$
$$\Rightarrow A_M = (A_\mu, A_i, B_j)$$

$$\lambda = \psi \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Initial Configuration:
- One 10D-gauge field ($A_\mathrm{M}$)
- One spinor ($\lambda$ 10D-gaugino)

Final Configuration:
- One 4D-gauge field ($A_\mu$)
- Six scalar fields ($A_i$ and $B_j$)
- Four spinors (4D-gauginos)

# Multiplet Structure

$A_i = Sa_i + \widehat{A_i}, \qquad B_j = Sb_j + \widehat{B}_j, \qquad a_i a_i + b_j b_j = 1$

$B_i^a = D_i S^a, \qquad B_i^a = \frac{1}{2}\epsilon_{ijk} F_{jk}^a$

$\psi_0^a = \gamma^0 M^{-1/2} (\boldsymbol{\sigma} \cdot \mathbf{B})\, u, \qquad Pu = u$

$P = -\gamma^0 \gamma_5 \alpha^i a_i + i\gamma^0 \beta^j b_j$

$\mathbf{S} = \frac{1}{4} u^\dagger \boldsymbol{\sigma}\, u \quad \Rightarrow \quad \mathbf{S} = \frac{1}{2} \sum_{n,s,s'} a_s^{n\dagger} (\boldsymbol{\sigma})_{ss'}\, a_{s'}^n$

| Sector | States | Spin Content |
|---|---|---|
| 0 excitations | 1 | $1\ \times$ (spin-0) |
| 1 excitation | 4 | $2\ \times$ (spin-$\frac{1}{2}$) |
| 2 excitations | 6 | $1\ \times$ (spin-1) + $3\ \times$ (spin-0) |
| 3 excitations | 4 | $2\ \times$ (spin-$\frac{1}{2}$) |
| 4 excitations | 1 | $1\ \times$ (spin-0) |
| Total | 16 | $1\ \times$ (spin-1) + $4\ \times$ (spin-$\frac{1}{2}$) + $5\ \times$ (spin-0) |

# Vanishing of the Beta Function

$$b_0 = \frac{11}{3} C_2(G) - \frac{2}{3} \sum_{\text{Weyl}} T(R_j) - \frac{1}{6} \sum_{\text{Real Scalars}} T(R_s),$$

$$\beta(g) = -\frac{g^3 b_0}{16\pi^2}$$

$C_2(SU(2)) = T(\text{adj}) = 2$

Four Gauginos

Six Scalars

$$b_0 = \frac{11}{3} \cdot 2 - \frac{2}{3}(4 \cdot 2) - \frac{1}{6}(6 \cdot 2) = \frac{22}{3} - \frac{16}{3} - \frac{6}{3}$$

$$\boxed{\beta(g) = 0}$$

# Montonen-Olive Duality

- Duality Statement
- Why $\mathcal{N} = 4$ model?
- Duality Group

# Duality Statement

$\mathcal{N} = 4$ super Yang-Mills theory with gauge coupling $g$ is physically equivalent to the theory with gauge coupling $4\pi/g$ with the roles of electrically and magnetically charged particles interchanged. Specifically, the W bosons of the weakly-coupled description are identified with the magnetic monopoles of the dual strongly-coupled description, and vice versa.

- Quantum theory compatible with the electric-magnetic symmetry of the classical theory
- $\mathcal{N} = 4$ is special in that charge quantization, multiplet structure, and exact BPS formula are simultaneously satisfied in this theory (for instance, breaks down in $\mathcal{N} = 2$)

# Duality Group

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D_\mu\phi^a)(D^\mu\phi^a) \quad \longleftarrow \quad *F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$$

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu\,a}F^a_{\mu\nu} - \frac{\theta g^2}{32\pi^2}F^{\mu\mu\,a*}F^a_{\mu\nu} - \frac{1}{2}D^\mu\phi^a D_\mu\phi^a$$

$$\mathcal{L} = -\frac{1}{32\pi}Im\left(\underbrace{\frac{\theta}{2\pi} + \frac{4\pi i}{g^2}}_{\tau}\right)\left(F^{\mu\nu\,a} + i*F^a_{\mu\nu}\right)\left(F^a_{\mu\nu} + i*\mathrm{F}^a_{\mu\nu}\right) - \frac{1}{2}D^\mu\phi^a D_\mu\phi^a$$

$T:\ \tau \mapsto \tau + 1$ ← Equivalent to changing θ by $2\pi$, which doesn't change $e^{i\,\theta k}$ (k is the topological charge of instantons)

$S:\ \tau \mapsto -\frac{1}{\tau}$ ← Equivalent to the exchange $g^2/4\pi \mapsto (g^2/4\pi)^{-1}$ (transformation proposed by Montonen-Olive)

⇒ S and T generate the duality group $SL(2,\boldsymbol{Z})$ which acts as a Möbius Transformation on $\tau$.

# Duality Group

$$\gamma:\ \tau \longmapsto \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \qquad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \longmapsto \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}, \qquad Q_m = \frac{4\pi}{g} n_m, \qquad Q_e = n_e e - n_m \frac{g\theta}{2\pi}$$

$$M^2 \geq 4\pi v^2 \begin{bmatrix} n_e & n_m \end{bmatrix} \frac{1}{\tau} \begin{bmatrix} 1 & -Re(\tau) \\ -Re(\tau) & |\tau|^2 \end{bmatrix} \begin{bmatrix} n_e \\ n_m \end{bmatrix}$$

Invariant under $SL(2, \boldsymbol{Z})$ transformations:

$$\boldsymbol{Z_2} \longrightarrow SL(2, \boldsymbol{Z})$$

# Testing the Duality

Topological Test by Vafa and Witten

# Vafa and Witten: Topological Test

**Idea:** Couple $\mathcal{N} = 4$ SYM to a curved background (transforms one supercharge Q into a nilpotent Lorentz scalar, $Q^2 = 0$)

$$S_{\text{twisted}} = \{Q, V\} + \frac{\tau}{8\pi i}\int_M \text{Tr}(F \wedge F) = \{Q, V\} + \frac{\tau\pi}{i}\text{k}$$

$$Z(M, \tau, G) = \sum_{k \geq k_0} \chi\left(\mathcal{M}_k(M, G)\right) q^k, \qquad q = e^{2\pi i \tau}$$

$$\boxed{Z\left(M, -\frac{1}{\tau}, G\right) = f(\tau, M) \cdot Z\left(M, \tau, \hat{G}\right)}$$

Purely Topological
(no G dependence)



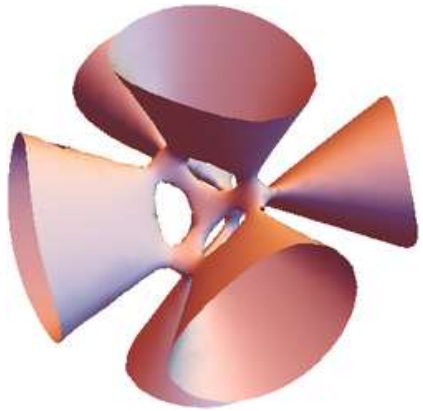
Expected by the duality

Recall: k is the topological charge of instantons.
We can write it as: $k = \frac{1}{8\pi^2}\int_M Tr(F \wedge F)$

# Vafa and Witten: K3 Test

Definitions:

- $\mathrm{G} = \mathrm{SU}(2)$
- $\mathrm{M} = \mathrm{K3}$ (unique simply-connected complex surface with trivial canonical bundle)
  - Invariants: $b_1 = 0,\ b_2^+ = 3,\ b_2^- = 19,\ \chi(K3) = 24,\ \sigma(K3) = -16$



Ex: Smooth quartic surface in 3-space. Picture shows part of the real points of a K3 surface with complex dimension 2

Instanton Moduli Space:

$$dim(\mathcal{M}_k = 8k - 3(1 - b_1 + b_2^+) = 8k - 12$$

⇒ No $\mathrm{SU}(2)$ instantons with $\mathrm{k} = 0$ or $\mathrm{k} = 1$

$$Z_{SU(2)}(K3, \tau) = \frac{1}{\Delta(\tau)} = \frac{1}{\eta(\tau)^{24}} = q^{-1}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{24}}$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

$$Z_{SU(2)}\left(K3, -\frac{1}{\tau}\right) = \underbrace{(-i\tau)^{-12}}\, Z_{SU(2)}(K3, \tau)$$

Determined purely by topology

# Further Checks and Conclusion

- String Theory
- Mathematical Structure
- Final Thoughts

# String Theory and more Math

The $\mathrm{SL}(2, \boldsymbol{Z})$ symmetry of the $\mathcal{N} = 4$ model is thought to relate to heterotic string theory on a six-torus.

- S-duality
- Some relation to $\mathrm{R} \leftrightarrow 1/\mathrm{R}$ from T-duality

S-duality compactified on a Riemann surface $\Sigma \Rightarrow$ geometric Langlands Correspondence

- D-modules on moduli stack of G-bundles on $\Sigma \Leftrightarrow$ local systems for Langlands dual group $\hat{G}$ on $\Sigma$
- Theory is related to fundamental mathematical structures (evidence + probing grounds)

# Conclusion

- Duality remains a conjecture, with no definitive proof

BUT

- Evidence from classical results
- Exact computations in $\mathcal{N} = 4$ SYM
  - Strong contender for manifestation of the duality
- Several mathematical tests of the theory

Restraint: Exact duality relies on the existence of magnetic monopoles

**First Results from a Superconductive Detector for Moving Magnetic Monopoles**

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(Received 5 April 1982)

A velocity- and mass-independent search for moving magnetic monopoles is being performed by continuously monitoring the current in a 20-$cm^2$-area superconducting loop. A single candidate event, consistent with one Dirac unit of magnetic charge, has been detected during five runs totaling 151 days. These data set an upper limit of $6.1 \times 10^{-10}$ $cm^{-2}$ $sec^{-1}$ $sr^{-1}$ for magnetically charged particles moving through the earth's surface.

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# Additional Resources

$\Gamma^{\mu} = \gamma^{\mu} \otimes 1 \otimes \sigma_3, \qquad \mu = 0,1,2,3$
$\Gamma^{3+i} = 1 \otimes \alpha^{i} \otimes \sigma_1, \qquad i = 1,2,3$
$\Gamma^{6+j} = \gamma_5 \otimes \beta^{j} \otimes \sigma_3, \qquad j = 1,2,3$
$\Gamma^{11} = 1 \otimes 1 \otimes \sigma_2, \qquad C = \mathcal{C} \otimes 1 \otimes 1$

$\{\alpha^i, \alpha^j\} = \{\beta^i, \beta^j\} = -2\delta^{ij}, [\alpha^i, \alpha^j] = -2\epsilon^{ijk}\alpha^k, [\beta^i, \beta^j] = -2\epsilon^{ijk}\beta^k, [\alpha^i, \beta^j] = 0$

$\mathcal{N} = 2$ model:

- Electric and magnetic multiplets no longer identical
- Non-zero β-function (inexact BPS formula)

Vafa-Witten Test:

- Non-perturbative (can get partition function over all topological factors without series expansion in g)
- Helps describe the relation between weak- and strong-coupling ($\mathrm{Im}\,\tau \to \infty$ versus $\tau \to 0$)
- Specific to $\mathcal{N} = 4$ (topological structure is related to the multiplet structure)