Finite Temperature Instantons in the $\mathbb{CP}^{N-1}$ model

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1 Overview / Summary

In this paper we explicate Ian Affleck's results on finite temperature gauge theories. Emphasis is placed on clarifying computations. We start by introducing topological terms in quantum field theory, how they arise, and the corresponding relevance of instantons. Next, we introduce finite temperature (equilibrium) quantum field theory, and we detail the standard ways in which quantum field theory changes. From there, we focus on the two-dimensional $\mathbb{CP}^{N-1}$ model,

deriving instantons and theta dependence of the free energy at both zero and finite temperatures. We emphasize the relevance of "quantum" instantons at nonzero temperatures, arising from the large n expansion, for the arisal of the topological term. Finally, we relate these concepts to physically relevant models (namely QED, which carries over results directly, and QCD).

2 Foundational Concepts

Below, we introduce the arisal of θ-terms in traditional quantum field theories, as well as the basic formalism of quantum field theory at finite temperature. Our eventual goal is to describe how the θ-dependence of the $\mathbb{CP}^{n-1}$ model arises as an instanton effect, but only at non-zero temperatures.

2.1 θ-Terms in QFT

In the original paper by Gross, Callan, and Dashen [3], from which most of the following is derived, they consider the vacuum expectation value of the time evolution operator:

$$\langle 0|e^{-tH}|0\rangle = \int \mathcal{D}A_\mu e^{-S}$$

where the integral only includes fields that go to pure gauge as $|x| \to \infty$. We enforce this condition in order to ensure that the action is finite, with the curvature tensor $F_{\mu\nu}$ vanishing far from the origin. In particular, this implies that the gauge field takes the form $A_\mu = g^{-1}\partial_\mu g$. Consider the "sphere at infinity", or a sphere of sufficiently large radius around the origin so that A_μ restricted to the sphere is pure gauge. We can construct a map from the sphere to the gauge group by taking the map sending $\phi \in S^{d-1}$ to $g \in G$ such that $A_\mu(x(\phi))$ takes the form $g(x(\phi))^{-1}dg(x(\phi))$ where $x(\phi)$ is the image of the circle in R^n. This defines a map $S^{d-1} \to G$; we know that maps of this form can be classified by the homotopy groups

$$\pi_{d-1} = \{f : S^{d-1} \to G\}/\sim$$

where the equivalence relation is homotopy equivalence. Using this construction, we can classify connections based on the homotopy group of their restriction to the sphere at infinity.

For example, consider a $U(1)$ gauge theory over two dimensional $d = 2$ space. Recall that $U(1) \simeq S^1$ as a manifold, so that maps $S^1 \to U(1) \simeq S^1$ are classified the number of times the image winds around the circle. This tells us that the first homotopy group of $U(1)$ is the

integers, which is of particular relevance for this paper. We want to distill this topological information of the field as a local quantity, a "topological charge" that we can integrate over all of spacetime. We can do so simply by employing Stokes theorem: note that the line integral of a one form is equivalent to the integral of a two form on the interior

$$\frac{1}{2\pi}\int_{\partial\mathbb{R}^2} A = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA = (\star)$$

where the "boundary" of $\mathbb{R}^2$ is the circle at infinity (really, we are considering the boundary of some large finite region). We divide out by 2π to get an integer value for the left-hand side of the equivalence: recall that any one-form on the sphere can be decomposed into its exact and nonexact components, so that $A = df + ad\theta$ for some function f and constant a. $d\theta$ is the "angle form", the canonical closed but not exact one form on the circle. Then

$$(\star) = \frac{1}{2\pi}a2\pi = a$$

The right hand side, in coordinates, gives us the local quantity

$$Q := \frac{1}{2\pi}\int d^2x\, q_\mu = \frac{1}{2\pi}\int d^2x\, \epsilon_{\mu\nu}\partial_\nu A$$

as the integrand. Classically, gauge theories involve some associated vector bundle over a base space - typically R^n. But given a topologically nontrivial base space, not all corresponding vector bundles are homotopic, meaning that the space of different field configurations is divided up into various topologically distinct sectors. Instantons then represent "tunneling" between various sectors. We can rewrite the integral $(\star)$ in terms of the difference of the integrals over different time slices of the action. then

$$\begin{aligned} Q &= \frac{1}{2\pi}\int_{\partial\mathbb{R}^2} A_\mu \\ &= \frac{1}{2\pi}\int dx A_1(\tau\to\infty) - \frac{1}{2\pi}\int dx A_1(\tau\to-\infty) \\ &=: n_\infty - n_{-\infty} \end{aligned}$$

which tells us that the topological charge at asymptotically different times has changed by some integer. We can physically interpret this as telling us that tunneling process between different topological sectors correspond to classical solutions with finite integer topological charge: this is the instanton. Define $|n\rangle$ to be the vacuum state with $Q = n$. Motivated by the above we can

see that tunneling between vacuum states is governed by the integral

$$\langle n|e^{-tH}|m\rangle = \int [\mathcal{D}A_\mu]_{(n-m)} e^{-S}$$

where the bracket notation is a restriction of the integral to connections in the bundle with $Q = (n - m)$. This tunneling probability tells us that $|n = 0\rangle$ is not a true ground state for the theory. We can see that each transition process, taking the semiclassical saddle point expression, has an amplitude proportional to

$$\langle n|e^{-tH}|m\rangle \sim O(e^{-(n-m)/g})$$

where g is some gauge coupling we've introduced. Classically, then, $|n\rangle$ to $|n\rangle$ processes dominate, contributing like $O(1)$. Perturbative calculations take $g \to 0$, meaning that instanton transitions go undetected by perturbative expansions. For non-vanishing coupling, however, they give a meaningful contribution to the action, and $|0\rangle$ is no longer a true vacuum.

In analogy to the Bloch states of condensed matter, we can construct a continuous family of true vacua. Start by considering some gauge transformation that changes the behavior at infinity, T. In particular, consider the operator sending $T|n\rangle = |n + 1\rangle$; such a gauge transformation must commute with the Hamiltonian, and so it must share its eigenstates. It also must be unitary, and so its spectrum must be contained in $U(1)$. We can define an eigenstate of T by

$$|\theta\rangle = \sum e^{in\theta} |n\rangle$$

for θ between 0 and 2π. Clearly

$$T|\theta\rangle = \sum e^{in\theta} |n + 1\rangle = e^{-i\theta} \sum e^{in\theta} |n\rangle = e^{-i\theta} |\theta\rangle$$

so $|\theta\rangle$ is invariant (up to scaling) under instanton processes. It is also stable under insertion of gauge invariant charge sources, and so these theta terms define a continuum of non-equivalent true ground states for the theory. These theta vacua are stable under small perturbation, so while $|\theta = 0\rangle$ has the lowest total energy, any one of these theories makes for a self-consistent (though physically distinct) solution around which perturbation theory can be performed. Generally, we fix one of these vacua for our computations by sending S to $S + i\theta Q$.

We find very similar structure in four dimensional Yang Mills theory (drawing from Tong's TASI lectures [7] and Haydys's PCMI notes for cleaner differential form notation [5]), with the relevance to QCD being the motivation behind much of classical gauge theory's development.

Here are the corresponding objects of study:

$$Q = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} d^4x \ \mathrm{tr}\, F_{\mu\nu}(*F)_{\mu\nu} = \frac{1}{6\pi^2}\int_{\partial\mathbb{R}^4} d^3x\, \epsilon_{ijk}\,\mathrm{tr}(A_iA_jA_k)$$

where the $8\pi^2$ comes from the first Chern class. In cleaner notation,

$$Q = \frac{1}{8\pi^2}\int \mathrm{tr}\, F\wedge F = \frac{1}{8\pi^2}\int \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$$

Notice that the expression on the right is the Chern Simons form, and it is also the exterior derivative of the expression on the left. The first term vanishes as a total derivative. In particular, note that for any N, the third homotopy group $\pi_3(SU(N)$ of $SU(N)$ is the integers, so vacua of these theories are classified by integers as well. The theta term of this model is physically relevant, theoretically breaking CP symmetry in the strong force.

Finally, we briefly introduce a trick to find the topological term for a given gauge theory: Bogomolny decomposition. Effectively, Bogolmony decomposition rewrites the Lagrangian of a system as the sum of a positive square and an extra term, giving a lower bound on the action. The extra term is the topological term discussed above, and minimizing configurations (instantons) arise as field configurations for which the positive square is zero. In the four dimensional case, we can do this nicely in coordinate free notation

$$\mathrm{tr}(F\wedge *F) = \frac{1}{2}\,\mathrm{tr}[(F\pm *F)\wedge *(F\pm *F)] \mp \mathrm{tr}(F\wedge F)$$

because $F\wedge *(*F) = F\wedge F$. The left hand side is the standard Lagrangian for Yang Mills, whereas the right hand side is some perfect square plus or minus the topological term. Then we can find an instanton for Yang Mills theory by minimizing, taking $F = \mp * F$ (a "self-dual" or "anti-self-dual" instanton. Below, we do the same thing more explicitly for the simpler $\mathbb{CP}^{N-1}$ model.

The $\mathbb{CP}^{N-1}$ model, is commonly used to study QCD, but it is an abelian gauge theory. As such, the relevant expressions will be mimick the expressions for two dimensions and gauge group $U(1)$ above. A lot of the model's dynamics, however, will mimick that of QCD. One particularly relevant feature is scale invariance, which introduces an arbitrary scale parameter into any derived instantons. The integral over this scale parameter does not generically converge, and so calculations related to instantons often require imposing infrared or ultraviolet cutoffs (with varied physical motivations). It is also the case that in the large N limit, at zero temperature, instanton effects become irrelevant. We will see here that finite temperature breaks scale invariance, and instantons (arising from a large N computation) make up the dominant

contribution to the θ dependence of the $\mathbb{CP}^{N-1}$ model.

2.2 QFT at Finite Temperature

Finite temperature QFT describes quantum field theory in the presence of a heat bath, particularly at equilibrium. Similar to how standard quantum field theory arises from imposing Lorentz invariance to quantum mechanics, we get thermal quantum field theory by doing the same for quantum statistical mechanics. In particular, we consider the Wick rotation of the statistical (relativistic) partition function, and we identify the Euclidean time with an inverse temperature τ. We integrate over this variable until we reach the true inverse temperature of the system, β giving us a partition function of form

$$Z := \operatorname{tr}\left[e^{-\beta H}\right] = \int \mathcal{D}\phi \exp\left(-\int_0^\beta d\tau \int d^d x\, \mathcal{L}[\phi]\right)$$

where ϕ is a stand-in for any dynamical variables in the system. A crucial feature of this partition function at equilibrium is that the trace restricts us to integrating over ϕ that are periodic or antiperiodic in Euclidean time. We'll see that we get periodicity for bosons and anti-periodicity for fermions. This can be seen more rigorously through the KMS conditions on operators, which I derive below: let's say we want to compute the correlation function of any two distinct operators, $A(t)$ and $B(t)$ with initial conditions A_0 and B_0. Computing correlators in finite temperature field theory transfers over nicely from statistical mechanics and standard QFT:

$$\langle A(t)B(t')\rangle = \frac{1}{Z}\operatorname{tr}\left[e^{-\beta H}A(t)B(t)\right]$$

we some simple manipulations reveal periodicity:

$$\begin{aligned}
\langle A(\tau)B(\tau')\rangle &= \frac{1}{Z}\operatorname{tr}\left[e^{-\beta H}A(\tau)B(\tau')\right] \\
&= \frac{1}{Z}\operatorname{tr}\left[e^{-\beta H}e^{-\tau H}A(0)e^{+\tau H}e^{-\tau' H}B(0)e^{+\tau H}\right] && \text{(Heisenberg evolution)} \\
&= \frac{1}{Z}\operatorname{tr}\left[e^{-(\tau+\beta)H}A(0)e^{+(\tau+\beta)H}e^{-\beta H}e^{-\tau' H}B(0)e^{+\tau' H}\right] \\
&= \pm\frac{1}{Z}\operatorname{tr}\left[e^{-\beta H}e^{-\tau' H}B(0)e^{+\tau' H}e^{-(\tau+\beta)H}A(0)e^{+(\tau+\beta)H}\right] && \text{(cyclicity of trace)} \\
&= \pm\frac{1}{Z}\operatorname{tr}\left[e^{-\beta H}B(\tau')A(\tau+\beta)\right]
\end{aligned}$$

We note that the using the cyclicity of the trace gives a minus sign when the operator is Fermionic, picking up a negative sign in the final result. This is the KMS relation; in particular, we can take $B(\tau')$ to be the identity, which gives us that any correlation function of an operator

A must be periodic (or anti-periodic) in its evolution over imaginary time, including propagators. The identity above is more commonly seen in its minkowski-time form, and taking $\tau' = 0$:

$$\langle A(t)B(0)\rangle = \langle B(0)A(t+i\beta)\rangle$$

Because time is periodic, we can treat our system as if it lives on $\mathbb{R}^d \times S^1_\beta$ (where the subscript denotes the radius), allowing us to decompose fields into their frequencies:

$$\phi(\tau) = \sum_n \phi(i\omega_n) e^{-i\omega_n \tau}$$

where the values of ω_n depend on the required periodicity:

$$\omega_n := \begin{cases} 2\pi n/\beta & \text{(Bosonic)} \\ 2\pi(n+1)/\beta & \text{(Fermionic)} \end{cases}$$

This is known as the Matsubara decomposition.

The new topology gives us one new way to detect topological structure: consider the Wilston loop that wraps around the space-time cylinder. We can consider the holonomy (or parallel transport) of any element of the vector bundle via the following operator

$$P(x_i) = \operatorname{tr}\left[\mathcal{P}\exp\left(\int_0^\beta d\tau\, A_0(x_i)\right)\right]$$

This is the Polyakov loop, which (across different classes of loops) serves as a representation of the homotopy group of the space, where the corresponding linear transformation is on the vector space fiber at the shared starting and ending point. Note that unlike the examples studied before, the topological nature of the constructed object did not arise from pure gauge in the large $|x|$ limit, but from the inherent nontrivial topology of the space.

3 The $\mathbb{CP}^{N-1}$ Model at $T = 0$

3.1 Introduction to the $\mathbb{CP}^{N-1}$ model

We now introduce the $\mathbb{CP}^{N-1}$ model, fleshing out the discussion in Affleck's paper [1]. The $\mathbb{CP}^{N-1}$ model, similar to the nonlinear sigma model, is a quantum field theory valued in a compact target manifold (namely $\mathbb{CP}^{N-1}$). It is an abelian gauge theory with an extra $SU(N)$

symmetry, described by the Lagrangian

$$\mathcal{L} = \frac{N}{g^2}|D_\mu n|^2$$

with n a vector in $\mathbb{C}^N$ constrained by $|n|^2 = 1$, and an implicit summation over μ. It has a $U(1)$ gauge field given by

$$A_\mu = \frac{i}{2}(n^*\partial_\mu n - n\partial_\mu n^*)$$

Note that for $\sigma : \mathbb{R}^2 \to \mathbb{R}$, the model is gauge invariant:

$$\begin{aligned} A_\mu &\mapsto \frac{i}{2}\left(e^{-i\sigma(x)}n^*\partial_\mu(e^{i\sigma(x)}n) - e^{i\sigma(x)}n\partial_\mu(e^{-i\sigma(x)}n^*)\right) \\ &= A_\mu + \frac{i}{2}(i\partial_\mu\sigma + i\partial_\mu\sigma)|n|^2 \\ &= A_\mu - \partial_\mu\sigma \end{aligned}$$

under gauge transformation, like a standard abelian connection.

It may not be a surprise that the $\mathbb{CP}^{N-1}$ model has many features in common with QED, and so many results derived below can be quickly transferred over. Despite being an abelian gauge theory, the $\mathbb{CP}^{N-1}$ model is also used as a testing ground for quantum chromodynamics, owing to three key features which both systems have in common. First, both theories are scale invariant, which is of particular importance to instanton computations. Second, both systems have asymptotic freedom, meaning that interactions weaken at high energy scales or small length scales. Finally, both exhibit confinement, with the n particles of the CP^{N-1} model coming in bound groups. The $\mathbb{CP}^{N-1}$ model is much easier to work with, making it a a nice testing ground for QCD-like behavior.

3.2 The topological term

Consider any configuration of the field such that the action is finite; we proceed in roughly the same manner as in the first section. To avoid divergence, we need that as $|x|$ becomes infinite, then $|D_\mu n| = 0$. We can unwrap this constraint, giving us that

$$0 = D_\mu n = \partial_\mu n + iA_\mu n$$

or equivalently that

$$A_\mu = n^*(A_\mu n) = i(n^*\partial_\mu n)$$

In particular, writing $n = n_i^{(0)} e^{i\sigma(x)}$ for constant $n^{(0)}$, we get that

$$A_\mu = i|n^{(0)}|^2(e^{-i\sigma}\partial_\mu e^{i\sigma}) = -\partial_\mu \sigma$$

meaning that A_μ approaches pure gauge for large x.

Given this constraint, we can consider the quantum field theory to be living on a topological two-sphere. $\mathbb{CP}^{N-1}$ bundles over S^2 can be classified by homotopy class by considering the integer value of a certain topological invariant, which we can find via Bogomolny factorization of the Lagrangian. First, define $D_\mp n = (D_1 \mp iD_2)n$, which look suggestively like the ∂ and $\overline{\partial}$ of complex analysis. We can Factor out the the square and retrieve our original Lagrangian plus a an extra term:

$$\begin{aligned}
|D_\mp n|^2 &= (D_1 n^\dagger \mp iD_2 n^\dagger)(D_1 n \pm iD_2 n) \\
&= |D_1 n|^2 + |D_2 n|^2 \mp i(D_1 n^\dagger D_2 n - D_2 n D_1 n^\dagger) \\
&= 2|D_\mu n|^2 \mp i\epsilon_{\mu\nu} D_\mu n^\dagger D_\nu n \\
&=: (2\mathcal{L} \mp iq)
\end{aligned}$$

or equivalently, that

$$\int d^4x\, |D_\mu n|^2 = \int d^4x\, [\frac{1}{2}|D_\mp n|^2 \pm \frac{i}{2}q]$$

We can identify the final term q as a topological charge density, and plugging in the connection A_μ it gives a total topological charge of

$$Q = \frac{\epsilon_{\mu\nu}}{2\pi} \int d^2x\, \partial_\mu A_\nu$$

We can then introduce a topologically nontrivial vacuum configuration for the model by adding a θ-term into the action, as described in the background paragraph.

$$S \mapsto S + i\theta Q = S + \frac{i\theta}{2\pi}\epsilon_{\mu\nu} \int d^2x\, \partial_\mu A_\nu$$

Our discussion will center around how the contribution of the θ term arises as an instanton effect at finite temperatures.

3.3 The zero temperature instanton

As discussed above, instantons are classical solutions to the equations of motion. Namely, instantons represent tunneling processes between topologicaly inequivalent sectors in the space of connections. We can use the Bogomolny decomposition above to find a simple instanton configuration for the connection:

$$D_{\mp} n = 0$$

this is our equivalent of the "(anti) self-dual" equations from the background section. The solutions found through this method do not exhaust all instantons, but they allow us to simplify the calculation of the first few.

Going beyond the scope of Affleck, our equations effectively force our field n to be (anti-)holomorphic (in analogy with ∂ or $\overline{\partial}$) but with respect to the gauge-covariant derivative instead of the conventional derivative [4]. Given some choice of sign, we can always fix the gauge so that these become equations of true holomorphicity. Then all Bogomolny instantons are equivalent to (anti-)holomorphic functions. Equivalently, we can pick coordinates on CP^{n-1}.

Based on the above, a generic instanton for the $T = 0$ model takes the form

$$n_\mu = \frac{w_\mu}{|w_\mu|}$$

for some arbitrary holomorphic function w_μ as a function of $z = x_1 - ix_2$. The generic solution can be written as

$$w_\mu = (s - s_0)u_\mu + \rho_m u$$

where u_μ parameterizes the $SU(N)$ pure gauge vacuum, and ρ_μ is the instanton term. Affleck shows that the instanton contribution at temperature zero is scale-dependent, diverging at large length scales. This will be clarified in the next section, applying large N methods.

3.4 Large N Expansion of the $\mathbb{CP}^{N-1}$ Model

Following Affleck's paper, we take the classic large N expansion of the $\mathbb{CP}^{N-1}$ model by imposing the constraint as a auxiliary field:

$$S[n, \lambda] = \int d^2x \frac{N}{g^2} |D_\mu n|^2 \mapsto \int d^2x \frac{N}{g^2} \left[|D_\mu n|^2 + i(|n|^2 - 1)\lambda(x) \right]$$

we can manipulate this expression into something quadratic in n, which we can quickly

integrate. We'll also introduce a mass parameter in the equation below.

$$\begin{aligned} S[n,\lambda] &= \frac{N}{g^2}\int d^2x\, n^\dagger(-D_\mu D_\mu + M^2 + i\lambda)n - i\frac{N}{g^2}\int d^2x\,\lambda(x) \\ &= N\,\mathrm{tr}\ln(D^2 + M^2 + i\lambda) - i\frac{N}{g^2}\int d^2x\,\lambda(x) \end{aligned}$$

giving us an effective action $S_{\text{eff}}[A_\mu, \lambda]$ independent of n. Assume that the action (without mass) has some stationary point (A_0, λ_0) around which we'd like to expand. Taking the point $A_\mu = 0, i\lambda = 0$ which we can ensure by adjusting the new mass parameter, we can extremize in M^2:

$$\begin{aligned} \frac{\delta S_{\text{eff}}}{\delta M^2} &= N\,\mathrm{tr}\left[\frac{1}{D^2+M^2}\right] - \frac{N}{g^2}\int dx \\ &= N\,\mathrm{vol}(\mathbb{R}^4)\int\frac{d^2k}{k^2+M^2} - \mathrm{vol}(\mathbb{R}^4)\frac{N}{g^2} = 0 \\ &\implies \int^\Lambda \frac{d^2k}{k^2+M^2} = \frac{1}{g^2} \end{aligned}$$

where we've imposed an ultroaviolet cutoff in the final line. Integrating, we find that

$$\frac{1}{2\pi}\ln\frac{\Lambda}{M} = \frac{1}{g^2}$$

Define a renormalization scale μ along with a renormalization constant g_R such that

$$\begin{aligned} g_R^{-2} &:= g^{-2} - \frac{1}{2\pi}\ln\frac{\Lambda}{\mu} \\ &= \frac{1}{2\pi}(\ln\frac{\Lambda}{M} - \ln\frac{\Lambda}{\mu}) \\ &= \frac{1}{2\pi}\ln\frac{\mu}{M} \end{aligned}$$

allows us to see that

$$M = \mu\exp\left(-\frac{2\pi}{g_R^2}\right)$$

To proceed, we can expand around this point in the following manner: take $H_0 = \partial^2 + M^2$, and

note that

$$\begin{aligned} -D^2 + M^2 &= \partial^2 + 2iA_\mu\partial_\mu + i(\partial_\mu A_\mu) - A_\mu A_\mu + M^2 \\ &= H_0 + V(A) \end{aligned}$$

Then we can expand (using the Taylor expansion of $\ln$) the trace as

$$\operatorname{tr}\ln(H_0 + V) \approx \operatorname{tr}\ln H_0 + \operatorname{tr} H_0^{-1}V - \frac{1}{2}\operatorname{tr}(H_0^{-1}VH_0^{-1}V)$$

to second order. We know that the terms in this expansion of each order must be gauge invariant; then the linear tadpole term must vanish, and the quadratic term must be proportional to $\operatorname{tr} F_{\mu\nu}F_{\mu\nu}$. We can find the proportionality constant by considering the corresponding quadratic diagrams, which are one loop corrections to the $U(1)$-gauge field propagator. From now on, we refer to the corresponding particle as a photon (slightly abusing terminology).

The corresponding contribution of the action must take the form

$$S^{(2)}_{\text{eff}} = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2} A_\mu(-p)\Pi_{\mu\nu}(p)A_\nu(p)$$

Gauge invariance forces $p_\mu\Pi_{\mu\nu} = 0$, or else the action picks up some nonzero integral of $A(-p)\Pi_{\mu\nu}(p)(ip_\nu\sigma)$ (plus a another cross term). Generically, $\Pi_{\mu\nu} = f(p)p_\mu p_\nu + g(b)g_{\mu\nu}$, but this constraint gives us that

$$f(p)p^2p_\nu + g(p)p_\nu = 0$$

or that generically,

$$\Pi_{\mu\nu} = \Pi(p)(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})$$

note that for $p \equiv 0$, the coefficient function simplifies to

$$S^{(2)}_{\text{eff}} = \frac{N}{48\pi M^2}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$$

via a standard loop integral. Note that here we equivalence of $\frac{1}{2}F_{\mu\nu}F_{\mu\nu}$ and $A_\mu(p^2g_{\mu\nu} - p_\mu p_\nu)A_\nu$, which is forced to be true by gauge invariance (or can be derived directly). This gives us an effective gauge theory in A_μ at large scales, making the theory reminiscent of QED. When we add in a theta term we get a $1/N$ expansion; but the manipulations above show us that instanton effects are irrelevant: to second order, A_μ is not constrained, though its derivatives are. Because of this, the system is not forced to be in pure gauge, and so topological sectors do not emerge from the system. We see next that this changes for non-finite temperatures.

To extract theta dependence at zero temperature, we appeal to the fact that the effective action constructed, to quadratic order in $1/N$, matches exactly with QED (a fact that will be clarified in the last section for the finite temperature case). We get the theta dependence of the temperature zero model from the background free energy, which can be taken exactly from QED (up to replacement of constants. In QED, adding a theta term corresponds to adding a constant background electric field. Properly substituting constants, we get that

$$F(\theta)/L = \frac{3}{2\pi N}(M\theta)^2 + O(N^{-2})$$

This form is not suggestive of instanton contributions, achievable directly from a $1/N$ power expansion. We can see that instanton contributions for the $\mathbb{CP}^{N-1}$ model don't contribute significantly at temperature zero. The action depended on derivatives of the connection, allowing the field to act arbitrarily badly at high energies. Thus, the effective S^2 topology broke down, since we don't have pure gauge to give us a nice map into the gauge group, and topological sectors become equivalent.

4 $\mathbb{CP}^{N-1}$ Model Instantons at Finite Temperatures

4.1 Instantons at Finite Temperature

At finite temperatures, the topology of the base space dramatically changes, drastically changing the results above. Recall from our earlier discussion that time becomes periodic, and changing the integral over p_0 into a Matsubara sum: Using Affleck's conventions, the momentum integral becomes

$$\int\int \frac{dk_0}{2\pi}\frac{dk_1}{2\pi} = \frac{1}{\beta}\sum_{k_0=2\pi n/\beta}\int \frac{dk_1}{2\pi}$$

This will be the key to unlocking a number of exciting effects of finite temperature.

4.1.1 Effective mass of photon

Note that a nontrivial mass arises at finite temperature. To start with, notice that Lorentz symmetry is broken, which has important consequences. Before, we had that Π vanished as $p_\mu p_\nu$ went to zero, which was necessary to ensure that the value at zero was not dependent on the direction of approach. Now, p_0 is split into discrete sectors; this allows us to construct Π

such that

$$\begin{cases} \Pi_{01}(0,p_1) = \Pi_{01}(0,p_1) = \Pi_{11}(0,p_1) = 0 \\ \Pi_{00} = m \end{cases}$$

for some constant m. This gives us a quadratic term depending on $M^2 A_0^2$, or in other words, the photon is allowed to have a mass.

As an aside, we can show the presence of such a nonzero mass using our expansion aboveit will serve as a useful first model for a common summation technique. Π comes from summing over the two one-loop vacuum polarization diagrams, including the one with two three-vertices and the one with one four-vertex. Consider a n particle entering from the top of the loop enters with momentum p and exits with momentum $p+k$. The vertices of the theory (coming from expanding the original Lagrangian) are

$$(V_3) = iA_\mu(n^\dagger \partial_\mu n - n\partial_\mu n^\dagger), \quad (V_4) = A_\mu A_\mu |n|^2$$

Taking the Fourier transform of both, we get the corresponding vertices:

$$V_3 = (2p_\mu + k_\mu), \;\; V_4 = 2g_{\mu\nu}$$

from here, we compute the loop term as in standard QED: summing over both bubbles, we get that

$$\Pi_{\mu\nu}(k) = \frac{N}{\beta}\int \frac{d^2p}{(2\pi)^2}\left(\frac{(2p+k)_\mu (2p+k)_\nu}{(p^2+m^2)(p^2+k^2+m^2)} - \frac{2g_{\mu\nu}}{p^2+m^2}\right)$$

where the β factor comes from the sum we're suppressing (in the integral), and the N from summing over all component n fields. We study, in particular, the value of $\Pi_{00}(0)$: recalling the summation over the first coordinate of p, which replaces the integral at finite temperature, we get

$$\begin{aligned} \Pi_{00}(0) &= \frac{N}{\beta}\int \frac{dp_1}{(2\pi)}\sum_{p_0}\left(\frac{4p_0^2}{(p_0^2+p_1^2+M^2)(p_0^2+p_1^2+M^2)} - \frac{2}{p_0^2+p_1^2+M^2}\right) \\ &= \frac{N}{\beta}\int \frac{dp_1}{(2\pi)}\sum_{n}\left(\frac{4(2\pi n/\beta)^2}{((2\pi n/\beta)^2+\omega^2)^2} - \frac{2}{(2\pi n/\beta)^2+\omega^2}\right) \\ &= \frac{N}{\beta}\int \frac{dp_1}{(2\pi)}\sum_{n}\left(\frac{2}{(2\pi n/\beta)^2+\omega^2} - \frac{4E^2}{((2\pi n/\beta)^2+\omega^2)^2}\right) \end{aligned}$$

This sum can be done relatively easily using contour integration. The exact derivation of each term in this sum will be down below for slightly different reasons, but for now we quote the

result:

$$\sum_n [(\frac{2\pi n}{\beta})^2 + \omega^2]^{-1} = \frac{\coth(\beta\omega/2)}{2\omega}$$

We can differentiate to find the second term, giving us that overall, after summing,

$$\Pi_{00}(0) = \frac{N\beta}{2}\int \frac{dp_1}{2\pi}[\sinh^2(\beta(p_1^2+M^2)/2)]^{-1} =: M$$

clearly the integrand is is greater than or equal to zero, and non-zero on a set of nonzero measure. This gives us a new, positive mass for our theory.

4.1.2 Instanton derivation

Now, the sum structure above allows us to rewrite the effective action. First, specialize to the gauge with $A_1 \equiv 0$. We get that

$$\begin{aligned} S_{\text{eff}} &= N \operatorname{tr} \ln(-D^2 + M^2) \\ &= \int_0^\beta d\tau \int dx \left[\frac{N}{\beta}\sum_n \int \frac{k_1}{2\pi} \ln\left((k_0+A_0)^2 + k_1^2 + M^2\right)\right] \end{aligned}$$

Recall that this is similar in form to the loop integral above, up to the logarithm. We can compute this integral via differentiation of that same integral:

$$\begin{aligned} I(\beta) &= \frac{1}{\beta}\sum_n \int \frac{dk_1}{2\pi}[(\frac{2\pi n}{\beta})^2 + k_1^2 + M^2]^{-1} \\ &=: \frac{1}{\beta}\int \frac{dk_1}{2\pi}\left(\sum_n [(\frac{2\pi n}{\beta})^2 + \omega^2]^{-1}\right) \end{aligned}$$

To start with, we explicitly do the summation. We can exploit that fact that $\cot(\beta z/2)$ has poles in the complex plane at $2\pi n/\beta$ with residue $2/\beta$ to rewrite the sum as a contour integral, and then we can shift the contour to evaluate:

$$\begin{aligned} I(\beta) &= \int \frac{dk_1}{2\pi}\frac{1}{4\pi i}\oint dz \frac{\cot(\beta z/2)}{z^2+\omega^2} \\ &= -\int \frac{dk_1}{2\pi}\frac{1}{2}\left(\frac{\cot(i\beta\omega/2)}{2i\omega} + \frac{\cot(-i\beta\omega/2)}{-2i\omega}\right) \\ &= \int \frac{dk_1}{2\pi}\frac{\coth(\beta\omega/2)}{2\omega} \end{aligned}$$

Note that $\coth(\beta\omega/2) \to 1$, remembdering that $\omega = k_1^2 + M^2$, gives us the zero temperature

integral, dubbed $I(\infty)$. But expanding $\coth$ in terms of exponentials, we get that

$$\coth(\beta\omega/2) = 1 + \frac{2}{e^{\beta\omega} - 1}$$

We can separate out the 1 as the $I(\infty)$ term, and subtracting from both sides, we get that

$$I(\beta) - I(\infty) = \int \frac{dk_1}{2\pi} \frac{e^{-\beta\omega}}{\omega(1 - e^{-\beta\omega})}$$

More generally, this expression tells us that we can expand out our expression in powers of the quantity $e^{-\beta M}$. Finally, we can use this to evaluate our log integral by taking a derivative. First, notice that the computations above are only changed by a shift for A_0, giving us the following expression:

$$\begin{aligned} I_{A_0}(\beta) &= -\int \frac{dk_1}{2\pi} \frac{1}{2\omega i} \left(\cot(\beta(i\omega - A_0)/2) - \cot(\beta(-i\omega - A_0)/2)\right) \\ &= \int \frac{dk_1}{2\pi} \frac{1}{\omega} \frac{\sinh(\beta\omega)}{\cosh(\beta\omega) - \cosh(\beta A_0)} \end{aligned}$$

Using the following identity:

$$\cot(a + ib) - \cot(a - ib) = -\frac{2i\sinh(2b)}{\cosh(2b) - \cosh(2a)}$$

We can integrate the inside expression in ω to get

$$\begin{aligned} V(A_0) &\propto \frac{1}{\beta} \int \frac{dk_1}{2\pi} \ln[\cosh(\beta\omega) - \cosh(\beta A_0)] \\ &= \frac{1}{\beta} \int \frac{dk_1}{2\pi} [\beta\omega + \ln(1 + \frac{\sin^2(\beta A_0/2)}{\sinh^2(\beta\omega/2)}] \end{aligned}$$

Properly tracking the constants, and discarding the A_0-independent constant, we get that

$$V(A_0) = \frac{N}{\beta} \int \frac{dk_1}{2\pi} \ln\left(1 + \frac{\sin^2(\beta A_0/2)}{\sinh^2(\beta(k_1^2 + M^2)/2)}\right)$$

This formula allows us to recover instanton effects! In particular, note that now the minima of the effective action lie at values where $\sin(\beta A_0/2)$ vanishes, or where $A_0 = 2\pi n/\beta$ for integer n. For finite action, then, we require that $A_0(x \to \pm\infty) \to 2\pi n_\pm/\beta$, where the $n_\pm$ may be distinct; we've recovered topological sectors, as well as finite action instantons tunneling between them. Below, we explore the consequences of adding a θ term to the effective action.

4.2 Low Temperature Theta Dependence as an Instanton Effect

We can define the topological charge as the difference $Q = n_+ - n_-$, drawing from the first chapter. Note that these solutions live on the imaginary time cylinder. We call this type of instanton arising from finite-temperature topology a "caloron"; in Affleck's paper, he refers to them as "quantum calorons", as they arise here from the large N expansion. We can add a term $i\theta Q$ to the action, shifting into a different topological sector, and find the theta dependence of the effective free energy via instantons. This is not easy at arbitrary temperatures, but there is a nice simplification that comes from low temperatures.

Recall from the previous section that the effective action can be expanded in powers of $e^{-\beta M}$. We can expand the potential to first order in $e^{-\beta M}$ and retain the gauge term that we derived in the Large N section, simplifying based on the gauge:

$$S_{\text{eff}}^{(2)} = \frac{N}{48\pi M^2}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu} \mapsto \frac{N}{24\pi M^2}\int (\partial_1 A_0)^2$$

We can Taylor expand the potential to first order and imagining βM is much larger than one:

$$\begin{aligned}
V(A_0) &= \frac{N}{\beta}\int \frac{dk_1}{2\pi}\ln\left(1 + \frac{4\sin^2(\beta A_0/2)}{(e^{\beta\omega/2} - e^{-\beta\omega/2})^2}\right) \\
&\approx \frac{N}{\beta}\int \frac{dk_1}{2\pi}\ln\left(1 + 4\sin^2(\beta A_0/2)e^{-\beta\omega}\right) \\
&= 4\sin^2(\beta A_0/2)\frac{N}{\beta}\int \frac{dk_1}{2\pi}e^{-\beta\omega} \\
&= 2(1-\cos(\beta A_0))\frac{N}{\beta}\int \frac{dk_1}{2\pi}e^{-\beta\omega} \\
&\approx [2e^{-\beta M}(M/2\pi\beta)^{1/2}]\frac{N}{\beta}(1-\cos(\beta A_0))
\end{aligned}$$

Defining the term in brackets as a constant ξ, we can finally express the low temperature potential:

$$S_{\text{eff}}[A_0] = N\int d\tau \int dx_1 \left[\frac{1}{24\pi M^2}(\partial_1 A_0)^2 + \frac{\xi}{\beta}(1-\cos(\beta A_0))\right]$$

Note that this is the action of the sine-Gordon system! our system has the same soliton solutions as this system. Note that the equation of motion enforces that $\partial_1 A_0$ is of order $O(e^{-\beta M/2})$, so any higher derivative term can be reasonably ignored at low temperatures. The one dimensional sine-Gordon equation has a standard kink soliton, satisfying

$$A_0(x) = \frac{4}{\beta}\arctan\left(\exp\left[\frac{12\pi M^2\xi}{\beta}(x-\overline{x})\right]\right)$$

for some shift $\overline{x}$ with topological charge

$$Q = \frac{\beta}{2\pi}(A_0(\infty) - A_0(-\infty)) = 1 - 0 = 1$$

Until now, we have been ignoring the impacts of the constraining field λ. It remains to show that the equation for λ is satisfied by our instanton solution.'The equation is

$$\frac{1}{-D^2 + M^2 + i\lambda} = \frac{1}{g^2}$$

We can expand this term in terms of A_μ to get the QED-like term:

$$\frac{1}{-D^2 + M^2 + i\lambda} - \frac{1}{48\pi^2 M^4} \operatorname{tr} F_{\mu\nu}F_{\mu\nu} + O(e^{-\beta M}) = \frac{1}{g^2}$$

But the higher order contributions can be ignored, as A_0 changes slowly (recall: $\partial_1 A_0$ is order $O(e^{-\beta M/2})$. This equation is solved by $i\lambda = 0$, and A_0 given by the sine-Gordon equation; our instanton is truly a solution.

Finally, we check that the instanton contribution is meaningful. By solving our sine-Gordon equation, we find that the instanton contribution to the action takes the form

$$S_0 = \frac{2N}{M}\sqrt{\frac{\xi}{3\pi\beta}}$$

Expanding out $\xi = 2e^{-\beta M}(M/2\pi\beta)^{1/2}$, we see that to get a contribution much greater than one, it is necessary that $Ne^{-\beta M/2} >> 1$, . Then combining with the low temperature limit, our predictions are reliable as an instanton effect in the parameter range

$$\frac{1}{N} << e^{-\beta M/2} << 1$$

Equivalently, the results are reliable when $1 << g^2 << N$ (as we might have expected).

We get from this, by standard quantum mechanics, the theta dependence of the free energy. Recall that we can write our system's partition function as

$$Z(\theta) = \sum_Q e^{iQ\theta} Z_Q$$

where Z_Q is a path integral restricted to classical paths with a certain topological charge. For

our model, we assume the dilute gas approximation of noninteracting instantons. In particular, can write Z_Q as a product of first order instanton contributions, and decompose $Q = n_+ - n_-$. The partition function for the system then takes the form

$$Z(\theta) = \sum_{n_\pm} \frac{1}{n_+! n_-!} (KL\beta e^{S_0 + i\theta})^{n_+} (KL\beta e^{-S_0 - i\theta})^{n_-}$$

where $L\beta$ is the area, and K is some correction from gaussian fluctuations. Separating the two sums, and converting them into exponentials, we get

$$\begin{aligned} Z &= \exp(KL\beta e^{-S_0 + i\theta}) \exp(KL\beta e^{-S_0 - i\theta}) \\ &= \exp(2KL\beta e^{-S_0} \cos(\theta)) =: e^{-F(\theta)} \end{aligned}$$

Finally, we recover from this the free energy based on (dilute-gas) caloron summation

$$\frac{F(\theta)}{L} := -2K \exp\left[-\frac{2N}{M} \left(\frac{\xi}{3\pi\beta} \right)^{1/2} \right] \cos(\theta)$$

This is the final result of Affleck's paper, constructing the theta dependence of the model from instanton contributions.

5 Extensions to Physical Systems

5.1 Analogy to QED at Finite Temperature

We can recover QED computations from this theory by taking $e^2 = 12\pi M^2/N$. The effective action for quantum electrodynamics, integrating out spinor fields, is

$$S_{\text{QED}} = \frac{1}{4e^2} \int d^2x \, \text{tr} \, F_{\mu\nu} + \text{tr} \ln(-D^2 + M^2)$$

In standard two dimensional QED, adding a theta term of the form above amounts to adding an external magnetic field.

From the derivation above, we see that the photon gains a mass at finite temperature. The corresponding effective action, as derived above, is

$$S_{\text{QED}} \approx \int_0^\beta d\tau \int dx_1 \left[\frac{1}{2e^2} (\partial_1 A_0)^2 + \frac{\xi}{\beta} (1 - \cos(\beta A_0)) \right]$$

We can derive from this the classical Coulomb gas. Consider the partition function independent of x_0:

$$\begin{aligned} Z &= \int \mathcal{D}A_0 \exp\left[-\beta \int dx_1 \left(\frac{1}{2e^2}(\partial_1 A_0)^2 + \frac{\xi}{\beta}(1-\cos(\beta A_0))\right)\right] \\ &= \int \mathcal{D}A_0 \exp\left[-\beta \int dx_1 \left(\frac{1}{2e^2}(\partial_1 A_0)^2\right)\right] \sum_{n_+,n_-=0}^{\infty} \frac{1}{n_+!n_-!}\left(\int dx_1 e^{i\beta A_0}\right)^{n_+}\left(\int dx_1 e^{-i\beta A_0}\right)^{n_-} \end{aligned}$$

taking $\rho = \sum_i^{n_+} \delta(x_1 - x_i^+) - \sum_j^{n_-} \delta(x_1 - x_j^-)$, we can rewrite this as

$$\begin{aligned} &= \int \mathcal{D}A_0 e^{-\int dx_1\left[\frac{\beta}{2e^2}(\partial_1 A_0)^2 - i\beta A_0 \rho\right]} \sum_{n_+,n_-=0}^{\infty} \frac{(\xi/2)^{n_+ + n_-}}{n_+!n_-!}\left(\Pi_i^{n_+}\int dx_i^+\right)\left(\Pi_j^{n_-}\int dx_j^-\right) \\ &= \sum_{n_\pm=0}^{\infty} \frac{(\xi/2)^{n_+ + n_-}}{n_+!n_-!}\left(\Pi_i^{n_+}\int dx_i^+\right)\left(\Pi_j^{n_-}\int dx_j^-\right)\exp\left[-\frac{\beta e^2}{4}\int dxdy\,\rho(x)|x-y|\rho(y)\right] \end{aligned}$$

where in the last step we integrated. This is the intended result. Here, ξ takes the role of the relativistic kinetic contribution of each particle to the partition function. Adding a theta term corresponds to adding in charges at opposite limiting values of x_1, setting up a background field.

5.2 Extensions to QCD

To conclude, we quote some results about quantum chromodynamics from Affleck's second paper [2]. Recall the shared properties between QCD and the $\mathbb{CP}^{N-1}$ model: asymptotic freedom, confinement, and scale invariance. At temperature zero, confinement allows theta dependence to be computed directly from a perturbative expansion. For example, a non-zero theta term can arise from a divergent four-point gluon interaction. Like for the $\mathbb{CP}^{n-1}$ model, this tells us that instanton contributions are likely unimportant at zero temperature. Beyond some temperature, QCD experiences Debye screening. Because of this, Affleck expected that the theta term could not be achieved by any power of $1/N$, and so Affleck expected that the theta term was an instanton effect. Affleck shows that the Theta depended free energy can be written in the following form:

$$F_T(\theta)/V = -\cos\theta N^{3/2} k \beta^4 e^{-S_{\text{eff}}}$$

which is very reminiscent of the result for the $\mathbb{CP}^{n-1}$ model, constructed from quantum calorons. He goes on to speculate that the theta term of QCD is built from quantum caloron contributions, as in the $\mathbb{CP}^{n-1}$ model.

While Affleck himself did not speculate this, the effective action of Yang Mills does indeed take

on a form similar to a sine-Gordon equation [6]. Polyakov derives the following from monopole contributions:

$$Z = \int \mathcal{D}\chi e^{-C} \exp\left[(\nabla\chi)^2 - M^2\cos(\chi)\right]$$

for some constant C. However, the dominant semi-classical contribution is made by monopoles and other semi-classical effect, and not calorons.

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