Scaling the Sphaleron: Finite Temperature Baryogenesis in the Electroweak Theory

Christopher Pezanosky

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1 Motivation

One of the most striking questions in particle physics arises from our everyday experience, which seems to afford many more encounters with particles than antiparticles. Namely, we find a glaring lack of proton-antiproton annihilations (very fortunately for our existence), the occurrence of which would dramatically alter our chemical composition as well as that of the objects around us. We might be enticed to believe that our Earth is somehow special and that the relative abundance of antimatter could be restored as we move beyond our planet to the scale of our galaxy. Indeed, cosmic rays offer us the first glimpse of antimatter (specifically 10^{-4} antiprotons for each proton) outside of high energy particle collisions [3]. However, this composition is *precisely* that which we expect from the secondary production process $p + p \to 3p + \bar{p}$ during the cosmic rays' travel towards us [15]. This is to say that there is *no* evidence of primordial antimatter in our galaxy.

As several authors have investigated [14, 13, 6], we could venture to ask what the lower bound is on the the scale over which baryon asymmetry is manifest. To answer this question, we note that, if large 'domains' of matter and antimatter existed in our universe, pair annihilations would naturally take place along the interfaces separating them [15]. If the domains (and thus the interfaces) were small enough, these annihilations would produce a diffuse γ-ray

background and consequently distort the cosmic microwave background. As this effect is not observed, the cited authors have concluded that the universe is completely composed of matter on all scales up to the Hubble radius. We may therefore safely claim that the observable universe is characterized by a definite asymmetry between matter and antimatter.

Our ability to quantify the relative abundance of antimatter comes almost entirely from models of primordial nucleosynthesis in the standard cosmology [15, 7]. In particular, this process provides predictions of the early abundances of all light elements, H,^{3} He,^{4} He, D, B, and ^{7}Li, from the single baryon-to-entropy ratio parameter

$$\eta \equiv \frac{n_B}{s} = \frac{n_b - n_{\bar{b}}}{s}. \tag{1}$$

Here n_b is the number density of baryons in the universe, $n_{\bar{b}}$ is the number density of antibaryons, n_B is their difference, and s is the entropy density of the universe. We may say that the calculation of the light element abundances are in agreement with observations if this parameter falls in the range

$$1.5 \times 10^{-10} < \eta < 7 \times 10^{-10}. \tag{2}$$

The dynamical production of the observed matter asymmetry parameter η is deemed 'baryogenesis.' While we will discuss the full (Sakharov) criteria for a proper theory of baryogenesis in the final section, it is clear that we require some mechanism of baryon number (B) violation as the starting point. The driving motivation for the following three sections will be to explain how this B-violating process can come about from the standard Electroweak Theory.

The unified theory of electromagnetic and weak nuclear interactions (i.e. Weinberg-Salam Theory) is described by the complete gauge group $SU(2)\times U(1)$ with corresponding Lagrangian density

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} + V(\phi) + \mathcal{L}_f, \tag{3}$$

where the field strengths are given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ W^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu. \end{aligned} \tag{4}$$

Here, B_μ is the $U(1)$ hypercharge gauge field and A^a_μ are the $SU(2)$ weak gauge fields. The complete covariant derivative of the Higgs field is

$$(D_\mu \phi)_i = \partial_\mu \phi_i - i[g A^a_\mu T^a + g' B_\mu Y]^j_i \phi_j, \tag{5}$$

where $Y = \frac{1}{2}\mathbb{I}$ is the hypercharge generator. The Higgs potential is modeled as

$$V(\phi) = \frac{\lambda}{4}(\phi^\dagger \phi - v^2)^2 \tag{6}$$

so that the gauge group is spontaneously broken down to the $U(1)$ group of electromagnetism today. Finally, $\mathcal{L}_f$ represents the fermionic sector of the theory.

We define the 'weak mixing angle' θ_W by the ratio of the two coupling constants,

$$\tan\theta_W \equiv \frac{g'}{g}, \tag{7}$$

with a measured value

$$\sin^2\theta_W \approx 0.23. \tag{8}$$

Exploiting the fact that the weak mixing angle is small, we may temporarily ignore the $U(1)$ hypercharge gauge field and analyze the zeroth-order $SU(2)$ Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(D_\mu\Phi)^\dagger D^\mu\Phi - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{\lambda}{4}\Big[\mathrm{Tr}(\Phi^\dagger\Phi) - v^2\Big]^2, \tag{9}$$

where we have considered only the bosonic sector of the theory. Using Trodden's notation [15], we have changed variables from the Higgs doublet $\phi = (\phi_1, \phi_2)$ to the matrix

$$\Phi(\mathbf{x}, t) = \begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix}, \tag{10}$$

such that

$$\Phi^\dagger\Phi = (\phi_1^*\phi_1 + \phi_2^*\phi_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

In this way, we may express the Higgs matrix Φ in terms of an $SU(2)$-valued field U:

$$\Phi = \frac{\sigma}{\sqrt{2}}U, \tag{12}$$

where $\sigma^2 = 2(\phi_1^*\phi_1 + \phi_2^*\phi_2) = \mathrm{Tr}\Phi^\dagger\Phi$, and U is uniquely defined at every spacetime point for which $\sigma(\mathbf{x}, t) \neq 0$.

From the requirement of a finite energy configuration, we demand that the energy density defined by the Lagrangian (9) vanish at infinity. Without loss of generality, we may therefore impose the conditions

$$\lim_{|\mathbf{x}|\to\infty} \sigma(\mathbf{x}, t) = v, \tag{13}$$

$$\lim_{|\mathbf{x}|\to\infty} U(\mathbf{x}, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

$\forall t$. (For energy to vanish, the Higgs must relax to its vacuum expectation value and the Higgs covariant derivative must vanish. We can choose the identity WLOG as a gauge choice that fixes the spatial gauge degrees of freedom. This will leave one final gauge dof that we will fix next).

In a vacuum configuration, $\mathrm{Tr}\Phi^\dagger\Phi = v^2$ and A^μ is pure gauge for all spacetime. In $A_0 = 0$ gauge, U is time-independent and we have

$$\begin{aligned} \Phi &= \frac{v}{\sqrt{2}}U \\ A_j &= \frac{1}{ig}\partial_j U U^\dagger \end{aligned} \tag{15}$$

For any time t at which $\sigma(\mathbf{x},t)$ is nonvanishing $\forall\mathbf{x}$, the Lie group-valued field $U(\mathbf{x},t)$ becomes a map from $\mathbb{R}^3$ with points at infinity identified (i.e. S^3) to $SU(2)$. But the mapping between S^3 and any simple, compact Lie Group G is classified by a 'winding number' $N \in \mathbb{Z}$. In our case, this implies $\pi_3(SU(2)) = \mathbb{Z}$, where configurations with distinct winding numbers cannot be smoothly deformed into each other. As a result, we have a manifest multiple vacuum structure, with each vacuum state characterized by the winding N_H of the Higgs field. We can write an expression for this integer-valued Higgs winding as

$$N_H(t) = \frac{1}{24\pi^2}\int d^3x\varepsilon^{ijk}\mathrm{Tr}[U^\dagger\partial_i UU^\dagger\partial_j UU^\dagger\partial_k U]. \tag{16}$$

Note that, in the vacuum configuration (15), the Higgs winding can also be identified with the winding of the $SU(2)$ gauge field. In fact, we can make another identification in this state by invoking the Chern-Simons current

$$K^\mu_{CS} \equiv \frac{g^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(A_\nu\partial_\rho A_\sigma + i\frac{2}{3}A_\nu A_\rho A_\sigma). \tag{17}$$

The 'Chern-Simons number' may then be expressed as

$$\begin{aligned} N_{CS}(t) &= \int d^3x K^0_{CS} \\ &= \frac{g^2}{32\pi^2}\int d^3x\varepsilon^{ijk}\mathrm{Tr}\bigg(A_i\partial_j A_k + \frac{2}{3}igA_iA_jA_k\bigg) \end{aligned} \tag{18}$$

Substituting in the pure gauge field A_μ from (15) in the vacuum configuration, this becomes

$$N_{CS}(t) = \frac{1}{24\pi^2}\int d^3x\varepsilon^{ijk}\mathrm{Tr}[U^\dagger\partial_i UU^\dagger\partial_j UU^\dagger\partial_k U] = N_H(t). \tag{19}$$

We therefore see that the integer-valued Higgs winding number of a vacuum configuration is equal to its Chern-Simons number. (Note, however, that a general (non-vacuum) configuration does not have integral Chern-Simons number).

We now reintroduce the fermion sector of the full Weinberg-Salam theory, with lepton and quark sectors (for a single family) described respectively by

$$\mathcal{L}_l = -i\bar\Psi\gamma^\mu D_\mu\Psi - i\bar e_R\gamma^\mu D_\mu e_R + h(\bar e_R\phi^\dagger\Psi + \bar\Psi\phi e_R), \tag{20}$$

$$\begin{aligned} \mathcal{L}_q = &- i\begin{pmatrix}\bar u & \bar d\end{pmatrix}_L\gamma^\mu D_\mu\begin{pmatrix}u\\ d\end{pmatrix}_L - i\bar u_R\gamma^\mu D_\mu u_R - i\bar d_R\gamma^\mu D_\mu d_R \\ &- Y_d\bigg[\begin{pmatrix}\bar u & \bar d\end{pmatrix}_L\begin{pmatrix}\phi^+\\ \phi\end{pmatrix}d_R + \bar d_R\begin{pmatrix}\phi^- & \phi^*\end{pmatrix}\begin{pmatrix}u\\ d\end{pmatrix}_L\bigg] \\ &- Y_u\bigg[\begin{pmatrix}\bar u & \bar d\end{pmatrix}_L\begin{pmatrix}-\phi^*\\ \phi^-\end{pmatrix}u_R + \bar u_R\begin{pmatrix}-\phi & \phi^+\end{pmatrix}\begin{pmatrix}u\\ d\end{pmatrix}_L\bigg], \end{aligned} \tag{21}$$

where again we have used the notation of [15], with $\phi = \begin{pmatrix}\phi^+ & \phi^0\end{pmatrix}$, $\phi^- = (\phi^+)^*$, and Ψ as the left-handed lepton doublet.

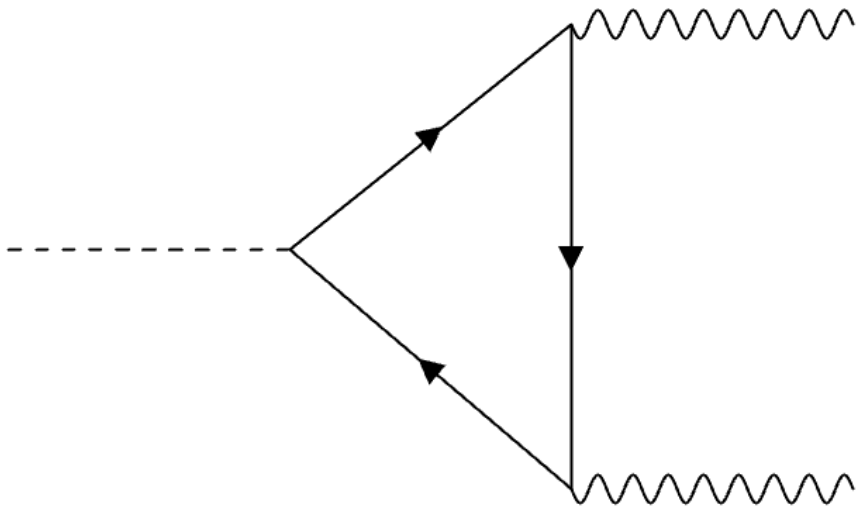

Figure 1: Adler-Bell-Jackiw ('triangle') diagram corresponding to the correlation function $\langle J_B^\mu(0) J_W^\alpha(x) J_W^\beta(y)\rangle$, which demonstrates the chiral anomaly mediated by left-handed quark fields coupling to the $SU(2)$ gauge fields A_μ^a.

We note that, classically, there exists a conserved global $U(1)$ current for each family, which seems to dictate the independent conservation of the number of each fermion species (i.e. 12 conserved currents in the Standard Model). In particular, within the quark sector, we may define a 'vectorlike' current

$$\begin{aligned} j_B^\mu &\equiv \frac{1}{3}(\bar{u}_L\gamma^\mu u_L + \bar{u}_R\gamma^\mu u_R + \bar{d}_L\gamma^\mu d_L + \bar{d}_R\gamma^\mu d_R) \\ &= \frac{1}{3}\Big\{(\bar{u}_R\gamma^\mu u_R + \bar{d}_R\gamma^\mu d_R) + (\bar{u}_L\gamma^\mu u_L + \bar{d}_L\gamma^\mu d_L)\Big\}. \end{aligned} \tag{22}$$

(with a natural generalization for all three families).

Likewise, we may define an axial current

$$\begin{aligned} j_A^\mu &\equiv \frac{1}{3}(-\bar{u}_L\gamma^\mu u_L + \bar{u}_R\gamma^\mu u_R - \bar{d}_L\gamma^\mu d_L + \bar{d}_R\gamma^\mu d_R) \\ &= \frac{1}{3}\Big\{(\bar{u}_R\gamma^\mu u_R + \bar{d}_R\gamma^\mu d_R) - (\bar{u}_L\gamma^\mu u_L + \bar{d}_L\gamma^\mu d_L)\Big\}. \end{aligned} \tag{23}$$

However, as cited by Trodden and originally noted by Adler, Bell and Jackiw [15, 2, 4], any axial current $\bar{\psi}\gamma^\mu\gamma^5\psi$ of a gauge coupled Dirac fermion ψ is anomalous. Indeed, as seen by the triangle diagram (Fig. 1), there is an anomaly in this axial current mediated by coupling to the $SU(2)$ gauge bosons. As we may observe, though, only the *left-handed* quark fields are charged under the weak

isospin fields (the right-handed fields are $SU(2)$ singlets). As a result, we may write the anomaly in j^μ_A as

$$\partial_\mu j^\mu_A = \cancel{\partial_\mu j^\mu_R} - \partial_\mu j^\mu_L = -\partial_\mu j^\mu_B. \tag{24}$$

Indeed, evaluating the triangle diagrams for both the baryon and lepton currents, we find

$$\partial_\mu j^\mu_B = \partial_\mu j^\mu_l = n_f \left(\frac{g^2}{32\pi^2} W^a_{\mu\nu} \tilde{W}^{a\mu\nu} - \frac{g_1^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right), \tag{25}$$

where n_f is the number of families, and

$$\begin{aligned} \tilde{F}^{\mu\nu} &= \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \\ \tilde{W}^{\mu\nu} &= \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} W_{\alpha\beta} \end{aligned} \tag{26}$$

are the duals of the $U(1)$ and $SU(2)$ field strength tensors.

As Klinkhamer and Manton cite [11], this anomaly may be written as a total divergence:

$$\partial_\mu j^\mu_B = \partial_\mu j^\mu_l = n_f \left(\frac{g^2}{32\pi^2} \partial_\mu K^\mu - \frac{g_1^2}{32\pi^2} \partial_\mu k^\mu \right), \tag{27}$$

where the (gauge-dependent) currents are defined as

$$\begin{aligned} K^\mu &= \varepsilon^{\mu\nu\alpha\beta} \left(W^a_{\nu\alpha} A^a_\beta - \frac{1}{3} g \varepsilon_{abc} A^a_\nu A^b_\alpha A^c_\beta \right) \\ k^\mu &= \varepsilon^{\mu\nu\alpha\beta} F_{\nu\alpha} B_\beta. \end{aligned} \tag{28}$$

Let us consider some trajectory in field configuration space beginning and ending in vacuum states (i.e. configurations satisfying (15) at $t = 0, t_f$). At these initial and final stages, the field strength averages to zero such that we may neglect the term involving k^μ (which is directly proportional to $F_{\nu\alpha}$). We therefore accumulate a total change in baryon number given by

$$\begin{aligned} \Delta B &\equiv \int_0^{t_f} dt\ \partial_0 \left(\int d^3x j^0_B \right) \\ &= n_f \frac{g_2^2}{32\pi^2} \left[\int dt\ \partial_0 \left(\int d^3x K^0 \right) + \int dt \int_S K^i \cdot dx_i \right]. \end{aligned} \tag{29}$$

In our choice of gauge where $U = \mathbb{I}$ at spatial infinity, $K^i = 0$ there as well, so that we obtain

$$\begin{aligned} \Delta B &= n_f \frac{g_2^2}{32\pi^2} \left[\int d^3x K^0 \right]_{t=0}^{t_0} \\ &= \frac{n_f}{3} \frac{g_2^3}{32\pi^2} \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abc} \left[\int d^3x A^a_\nu A^b_\alpha A^c_\beta \right]_{t=0}^{t_0}. \end{aligned} \tag{30}$$

However, we may invoke the condition that the field strength is zero in the vacuum to write

$$\begin{aligned} W^b_{\nu\alpha} &= \partial_\nu A^b_\alpha - \partial_\alpha A^b_\nu + g\varepsilon^{bcd} A^c_\nu A^d_\alpha = 0 \\ \Rightarrow \varepsilon^{\mu\nu\alpha\beta} A^a_\mu \partial_\nu A^b_\alpha &= -\frac{1}{2} g\varepsilon^{\mu\nu\alpha\beta}\varepsilon^{bcd} A^a_\mu A^c_\nu A^d_\alpha, \end{aligned} \tag{31}$$

which recovers the exact expression for the Chern-Simons number (18). We may thus identify the change in baryon number during a path between vacua as

$$\Delta B = \Delta N_{CS} \equiv n_f \Big(N_{CS}(t_f) - N_{CS}(0) \Big). \tag{32}$$

As N_{CS} is an integer for both the initial and final states, we find that transitions involving a change in Chern-Simons number also change the baryon number by integral multiples of the number of families n_f. To this end, Klinkhamer and Manton write for the classical Weinberg-Salam theory that 'if the fields traverse a noncontractible loop in configuration space, starting and finishing at the vacuum, with time as the parameter, then the baryonic charge changes by an integer' [11]. (They note that this integer is the Pontryagin index of the time-dependent gauge field).

Although the transitions between vacua indexed by N_{CS} generate an anomalous change in B, this is ultimately a purely *quantum* feature, as baryon number is still an exact global symmetry of Weinberg-Salam theory. To this effect, from our familiarity with the vacuum structure and CP-violating sector of QCD, we naturally expect an analogous quantum tunneling mechanism to occur in the Electroweak sector. As discussed in lecture with Professor Shenker, such *instantons* represent classical saddle-point solutions of the full four-dimensional Euclidean action mediating between degenerate vacua. The amplitude for such transitions to occur was consequently found to be proportional to

$$\exp(-S_E^{\text{instanton}}). \tag{33}$$

However, as we also saw during lecture, when the theory being considered has a Higgs field, there are a *family* of instanton soutions, rather than a single unique configuration that extremizes the Euclidean action. This leads to an instanton action $S_E^{\text{instanton}}$ which is not well-defined. As a result, we must fix the size of the Euclidean configurations in the functional integral. The action is then evaluated for such 'constrained instantons' before being integrated over instanton size.

As a solution for our required mechanism of B violation to construct a theory of baryogenesis, we might be tempted to invoke such instanton configurations as the obvious choice. Indeed, at zero temperature, this is the only viable means of anomalous baryon production. We will therefore sketch a quick estimate of whether instanton effects could supply the baryon asymmetry we are after. For this calculation, we repeat the procedure of 't Hooft (also cited by Trodden and discussed for the $U(1)$ gauge theory in lecture) [1, 15]:

We consider an instanton mediating between adjacent vacua of a pure gauge $SU(2)$ theory such that the change in Chern-Simons number $\Delta N_{CS} = 1$. We

note that the object

$$\int d^4x (W^a_{\mu\nu} - \tilde{W}^a_{\mu\nu})^2 \tag{34}$$

is positive semi-definite for any configuration of the $SU(2)$ gauge fields. Expanding this expression gives

$$\int d^4x \Big[\mathrm{Tr}(W_{\mu\nu} W^{\mu\nu}) + \mathrm{Tr}(\tilde{W}_{\mu\nu} \tilde{W}^{\mu\nu}) - 2\mathrm{Tr}(W_{\mu\nu} \tilde{W}^{\mu\nu}) \Big] \geq 0. \tag{35}$$

We may identify

$$\begin{aligned} \int d^4x \mathrm{Tr}(W_{\mu\nu} W^{\mu\nu}) &= 2g_2^2 S_E, \\ \int d^4x \mathrm{Tr}(W_{\mu\nu} \tilde{W}^{\mu\nu}) &= \frac{16\pi^2}{g_2^2} N_{CS} \end{aligned} \tag{36}$$

to obtain the bound on the Euclidean action

$$4S_E - 2\left(\frac{16\pi^2}{g^2}\right) N_{CS} \geq 0, \tag{37}$$

or, equivalently,

$$S_E \geq \frac{8\pi^2}{g^2} N_{CS}. \tag{38}$$

As Trodden notes [15], this bound is saturated by the $N_{CS} = 1$ insanton, such that the rate of B-violating instanton processes (i.e. B-violation at zero temperature) becomes

$$\Gamma_{\text{instanton}} \approx \exp(-2S_E) \approx 10^{-170}. \tag{39}$$

We remark that this rate is so suppressed that, if anomalous baryon production were only mediated by instantons, we would not have had a single B-violating event in our Hubble volume in the entire history of the universe [15]!

2 Pedagogical Introduction to Sphalerons

We have concluded that quantum tunneling through the high potential barrier separating N vacua is too inefficient to generate the necessary baryon asymmetry in our universe. In other words, baryogenesis at zero temperature, where instantons represent the only noncontractible loops passing through the vacuum configuration, is not possible. Since traveling 'through the barrier' is impractical, though, a natural extension is to consider going 'over the barrier.' Such classical transitions become possible at finite temperature due to 'kicks' from thermal excitation that surpass a certain threshold energy at the peak between degenerate vacua.

More formally, adjacent vacua are separated by a ridge of finite-energy configurations above the vacuum. The configuration of lowest energy on the ridge is

a static saddle-point solution of the Lorentzian action with one negative eigenvalue, and is referred to as the 'sphaleron' [11]. It is precisely the energy of this sphaleron that indicates the threshold which must be attained from thermal activation in order to interpolate between N_{CS} vacua and thus generate B-violating events.

To explain conceptually how sphalerons dictate the rate of transitions between vacua, we begin with a mechanical example borrowed from Trodden [15]: Consider an ideal pendulum confined to move within a plane, governed by the Lagrangian

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta). \tag{40}$$

The vacuum structure of this theory is periodic, labeled by

$$\theta_n = 2n\pi, \qquad n \in \mathbb{Z}. \tag{41}$$

The corresponding Schrödinger equation for the quantum analogue of this system takes the form

$$\frac{\hbar\omega}{2}\left(- \alpha\frac{d^2}{d\chi^2} + \frac{1}{\alpha}\sin^2\chi \right)\psi_n(\chi) = E_n\psi(\chi), \tag{42}$$

with $\chi \equiv \theta/2, \omega \equiv g/l$, and $\alpha \equiv \hbar\omega/(4mgl) \ll 1$.

Due to the periodicity of the potential, the energy eigenfunctions will also be periodic with an evident multiple vacuum structure. However, if the pendulum's energy is much less than the height of the barrier for transitions between vacua (i.e. the sphaleron), we can Taylor expand about a given minimum (e.g. $\theta = 0$), and the Schrödinger equation becomes that of a simple harmonic oscillator (tunneling between vacua is exponentially suppressed).

We can also imagine instead raising the temperature of the system by coupling the pendulum to a thermal bath. In this case, the system will be thermally excited to states of higher energy, and the pendulum will make spontaneous transitions between vacua over the barrier (crossing $\theta = \pm\pi$). The rate of these transitions will be determined by the height, or more precisely, the gravitational potential energy E_{bar}, of the barrier, such that

$$\Gamma_{\text{pendulum}}(T) \approx \exp\left(- \frac{E_{\text{bar}}}{T} \right). \tag{43}$$

In other words, for every sequence of configurations mediating between vacua, there is one configuration which maximizes the free energy, creating a family of these 'maximal energy configurations' for the full set of paths. Within this family, there is one configuration with the smallest free energy (i.e. saddle point), and it is the energy of this configuration that determines the transition rate over the barrier. For the Electroweak theory, this saddle point is the sphaleron.

3 Electroweak Sphaleron

We now replicate Klinkhamer and Manton's original analytic construction of the sphaleron for the Weinberg-Salam theory [11]. Borrowing from their approach, we will exploit the small magnitude of the weak mixing angle θ_W to approximate the theory to lowest order by its $SU(2)$ and Higgs sectors. Applying Klinkhamer's ansatz for the field configurations, we will obtain an explicit static solution to the classical field equations minimizing the energy functional. We then verify that the resulting object satisfies the relevant properties of the saddle-point solution we expect. Finally, we demonstrate that the configuration carries fractional baryon charge, justifying our physical interpretation of sphalerons as residing 'between' vacua. (Note that, while the authors highlight that the $SU(2)$ solution can be used to compute the sphaleron of the full electroweak theory by perturbative analysis in θ_W, we will concern ourselves strictly with the zeroth-order result, as this will supply the dominant contribution to the sphaleron energy relevant for baryogenesis).

Because the object we are after is a saddle-point of the energy landscape between vacua, we will look for a *static*, classical field configuration of the theory. To do this, we will set both the fermion fields and the time component of the gauge fields to zero. With these conditions, the energy functional corresponding to our Electroweak Lagrangian (3) becomes

$$E = \int d^3x \left[\frac{1}{4} W^a_{ij} W^a_{ij} + \frac{1}{4} F_{ij} F_{ij} + (D_i \phi)^\dagger (D_i \phi) + V(\phi) \right], \tag{44}$$

where the Higgs covariant derivative is defined by (5). The corresponding field equations are

$$(D_j W_{ij})^a = -\frac{1}{2} ig \left[\phi^\dagger \sigma^a D_i \phi - (D_i \phi)^\dagger \sigma^a \phi \right], \tag{45}$$

$$\partial_j F_{ij} = -\frac{1}{2} ig' \left[\phi^\dagger D_i \phi - (D_i \phi)^\dagger \phi \right], \tag{46}$$

and

$$D_i D_i \phi = 2\lambda \left(\phi^\dagger \phi - \frac{1}{2} v^2 \right) \phi, \tag{47}$$

where the field strength covariant derivative is defined as

$$(D_i F_{ij})^a = \partial_j F^a_{ij} + g \varepsilon^{abc} A^b_j F^c_{ij}. \tag{48}$$

Setting the weak mixing angle θ_W to zero, the $U(1)$ gauge field decouples and can be neglected from our analysis.

For this calculation, we will work in a different gauge from our initial boundary conditions (13,14), where the $SU(2)$-valued field $U(x)$ was set to the identity at infinity. Recall that this was done WLOG using our freedom to fix the spatial gauge degrees of freedom. Here, we instead work in a gauge such that

$$U^\infty \equiv \lim_{|\mathbf{x}| \to \infty} U(\mathbf{x}, t) = \frac{1}{r} \begin{pmatrix} z & x + iy \\ -x + iy & z \end{pmatrix} \tag{49}$$

Given that the conditions (15) must be satisfied at infinity for *any* configuration (i.e. the gauge fields must go to pure gauge), a plausible ansatz for the fields is of the form

$$\begin{aligned} A_i^a \sigma^a dx^i &= -\frac{2i}{g} f(gvr) dU^\infty (U^\infty)^{-1}, \\ \phi &= \frac{v}{\sqrt{2}} h(gvr) U^\infty \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{50}$$

(This is in fact very similar to the ansatz we made in class for the instanton in $U(1)$ gauge theory with a Higgs. Namely, we modulate the field boundary conditions by a spatially varying function).

Following Klinkhamer's notation, we define the dimensionless radial coordinate $\xi \equiv gvr$. Inserting our ansatz into the energy functional (44), we obtain

$$\begin{aligned} E = \frac{4\pi v}{g} \int_0^\infty d\xi \Bigg[4\left(\frac{df}{d\xi}\right)^2 + \frac{8}{\xi^2}[f(1-f)]^2 \\ + \frac{1}{2}\xi^2 \left(\frac{dh}{d\xi}\right)^2 + [h(1-f)]^2 + \frac{1}{4}\left(\frac{\lambda}{g^2}\right)\xi^2 (h^2-1)^2 \Bigg], \end{aligned} \tag{51}$$

with corresponding field equations that now take the form

$$\begin{aligned} \xi^2 \frac{d^2 f}{d\xi^2} &= 2f(1-f)(1-2f) - \frac{\xi^2}{4} h^2 (1-f), \\ \frac{d}{d\xi}\left(\xi^2 \frac{dh}{d\xi}\right) &= 2h(1-f)^2 + \frac{\lambda}{g^2}\xi^2 (h^2-1)h. \end{aligned} \tag{52}$$

We impose the following boundary conditions on our ansatz functions f, h:

$$\lim_{\xi \to 0} f = \alpha \xi^2, \qquad \lim_{\xi \to 0} h = \beta \xi \tag{53}$$

$$\lim_{\xi \to \infty} f = 1 - \gamma \exp\Big(-\frac{1}{2}\xi\Big), \qquad \lim_{\xi \to \infty} g = 1 - \frac{\delta}{\xi} \exp\Big(-\sqrt{\frac{2\lambda}{g^2}}\xi\Big) \tag{54}$$

The first set of conditions are required for regularity of the fields and their energy densities at the origin. (Powers of ξ compensate inverse powers of r in the ansatz). Meanwhile, the second set of conditions are obtained by demanding that the functions approach 1 at infinity (to recover our field boundary conditions), with a deviation from vacuum that decays with the relevant screening factor. In particular, the characteristic screening for the $SU(2)$ gauge field scales as

$$M_W \cdot r = \left(\frac{1}{2} gv\right)\left(\frac{\xi}{gv}\right) = \frac{\xi}{2}, \tag{55}$$

while the screening of the Higgs field is governed by the its mass

$$M_H \cdot r = (\sqrt{2\lambda} v)\left(\frac{\xi}{gv}\right) = \frac{\sqrt{2\lambda}}{g}\xi \tag{56}$$

(The extra factor of $1/\xi$ for the Higgs BC is tied to the spherical wave solution of the Klein Gordon equation).

The solution of (52) solves the field equations and thus minimizes the energy functional (44). As Klinkhamer notes, the instability of this solution pertains to field fluctuations outside the form of his ansatz. Moreover, by extrapolating our solution for different values of λ, we find that the energy of the solution increases with λ.

We can approximate the form of the sphaleron as well as its energy by a variational approach. Namely, we determined that the exact solution may be written in terms of the radial functions f and h minimizing the energy functional. We thus use a further ansatz for these functions that is parameterized by two free scale variables and has the correct limiting behavior as $r \to 0, \infty$. We then minimize the energy with respect to these parameters to approximate the true solution and bound the sphaleron energy.

A rough order of magnitude estimate for the energy can be obtained from the simple ansatz ('Ansatz a')

$$\begin{aligned} f^a(\xi) &= \begin{cases} \left(\frac{\xi}{\Xi}\right)^2, & \xi \le \Xi \\ 1, & \xi \ge \Xi, \end{cases} \\ h^a(\xi) &= \begin{cases} \frac{\xi}{\Omega}, & \xi \le \Omega \\ 1, & \xi \ge \Omega, \end{cases} \end{aligned} \tag{57}$$

for which the energy (44) becomes

$$E^a = \frac{4\pi v}{g}\frac{1}{210}\left\{\frac{1248}{\Xi} + \left[112 - 105\left(\frac{\Omega}{\Xi}\right) + 56\left(\frac{\Omega}{\Xi}\right)^3 - 12\left(\frac{\Omega}{\Xi}\right)^5\right]\Xi + 4\left(\frac{\lambda}{g^2}\right)\Omega^3\right\}. \tag{58}$$

We may then determine the scale parameters Ξ, Ω that minimize (58) as a function of λ/g^2.

In reality, though, Ansatz a is too rough, as it does not capture the deviation from vacuum that scales exponentially with the appropriate screening factor for each field, as we highlighted in our boundary conditions (53,54). In keeping with these conditions, we also implement the more robust 'Ansatz b'

$$\begin{aligned} f^b(\xi) &= \begin{cases} \frac{\xi^2}{\Xi(\Xi+4)}, & \xi \le \Xi \\ 1 - \frac{4}{\Xi+4}\exp[\frac{1}{2}(\Xi - \xi)], & \xi \ge \Xi, \end{cases} \\ h^b(\xi) &= \begin{cases} \frac{\sigma\Omega+1}{\sigma\Omega+2}\frac{\xi}{\Omega}, & \xi \le \Omega \\ 1 - \frac{\Omega}{\sigma\Omega+2}\frac{1}{\xi}\exp[\sigma(\Omega - \xi)], & \xi \ge \Omega, \end{cases} \end{aligned} \tag{59}$$

where $\sigma \equiv \sqrt{2\lambda/g^2}$. In addition to the correct boundary behavior, these functions and their first derivatives are now continuous at the interfaces $\xi = \Xi, \Omega$.

In units of $4\pi v/g$, the numerically optimized values of Ω and Ξ correspond to energies spanning 1.566 and 2.722 for λ/g^2 equal to 0 and ∞, respectively.

Using the experimental value $5\pi v/g \approx 5$ TeV, Klinkhamer and Manton obtain an estimate for the sphaleron energy between 7.9 TeV for $\lambda = 0$ and 13.7 TeV for $\lambda = \infty$. Today, we know from precision measurement of the Higgs boson mass that $\lambda \approx 0.129$, so that the sphaleron energy becomes $E_{\text{sph}} \approx 9.1$ TeV.

We can compare this analytic estimate to the energy obtained by numerically solving the field equations (52):

$$E^{\text{num}}(\lambda = 0) = 1.52\frac{4\pi v}{g} \approx 7.6 \text{ TeV}. \tag{60}$$

This is 3% lower than the Ansatz b result, which implies that the true sphaleron energy (at $\lambda \approx 0.129$) is a few percent below our estimate. We can also use this numerical result to confirm that the sphaleron (at $\lambda = 0$) is the maximal-energy configuration along a noncontractible loop in the space of field configurations and thus unstable. To do this, the authors invoke Manton's 1983 sphaleron analysis, which constructs a loop parameterized by the angle $\mu \in [0, \pi]$ starting and ending at the vacuum and passing through our sphaleron ansatz at $\mu = \pi/2$ [12]. The energy is obtained along this loop as

$$E^{\text{num}}(\lambda = 0, \mu) = \frac{4\pi v}{g}(1.28 \sin^2 \mu + 0.24 \sin^4 \mu). \tag{61}$$

We thus see that the energy indeed starts at zero, reaches its maximum at $\mu = \pi/2$, and then falls back to zero.

We now turn to computing the baryon charge of the sphaleron configuration. Since the sphaleron lies at the crest of the energy barrier between adjacent vacua with Chern-Simons number varying by one, we expect (assuming things are symmetric) that the configuration should carry baryon charge $\frac{1}{2}$. To verify this, we repeat the steps of the same calculation performed in (29), where we now consider a path starting in the trivial vacuum ($B = 0$) when $t = -\infty$ and arriving at the sphaleron configuration when $t = t_0$:

$$\begin{aligned} \Delta B = B_{\text{sph}} = \frac{g^2}{32\pi^2} \int_{-\infty}^{t_0} dt \int_S K^i \cdot dx_i \\ + \frac{g^2}{32\pi^2} \int_{t=t_0} d^3x K^0. \end{aligned} \tag{62}$$

Here, the gauge-dependent current K^μ is defined by (28) and S is the sphere at spatial infinity.

At spatial infinity, A^a_μ is pure gauge,

$$A^a_\mu \sigma^a dx^\mu = -\frac{2i}{g}(dU)U^{-1}, \tag{63}$$

and, as we noted in (14), we can without loss of generality always work in a gauge where $U = \mathbb{I}$ there. In this gauge, $K^i = 0$ at infinity, so that we obtain

$$B_{\text{sph}} = \frac{g^2}{32\pi^2} \int_{t=t_0} d^3x K^0. \tag{64}$$

It is manifest from this expression that the baryon charge depends only on the sphaleron configuration (i.e. the configuration at $t = t_0$) and not on the path traversed to attain it. Additionally, we see from the expression for the baryon current anomaly (25) that $\Delta B = B_{\text{sph}}$, and thus the path independence of this quantity, is gauge invariant.

However, we cannot simply plug our sphaleron ansatz (50) into the expression above, as we noted that this expression was derived in a different gauge, namely one in which $K^i = 0$ at spatial infinity. For this to be valid, we need our gauge field to fall off faster than $1/r$. To achieve this requirement from our ansatz gauge (49), we first perform a gauge transformation defined by

$$\tilde{U}^\infty = U_L U^\infty U_R \tag{65}$$

with

$$U_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U_R = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{66}$$

to obtain

$$A_i^a = -\frac{2f(gvr)}{gr^2}\varepsilon_{iab}x_b. \tag{67}$$

We now perform a second gauge transformation of the form

$$\mathbf{U}(\vec{x}) = \exp\left[\frac{1}{2}i\Theta(r)\vec{\sigma}\cdot\hat{\vec{x}}\right], \tag{68}$$

where $\Theta(r)$ spans from 0 to π as r increases from 0 to ∞. This modifies A_i^a to

$$\begin{aligned} A_i^a =& \frac{[1-2f(gvr)]\cos\Theta(r)-1}{gr^2}\varepsilon_{iab}x_b \\ &+ \frac{[1-2f(gvr)]\sin\Theta(r)}{gr^3}(\delta_{ia}r^2 - x_i x_a) \\ &+ \frac{1}{g}\frac{d\Theta}{dr}\frac{x_i x_a}{r^2}. \end{aligned} \tag{69}$$

Choosing $\Theta(r)$ to approach π fast enough as $r \to \infty$, this gauge field now goes to zero faster than $1/r$. The integral of K^i will thus now vanish at spatial infinity, and we can substitute K^0 calculated in this gauge into the expression (64) to obtain a baryon charge

$$B_{\text{sph}} = \frac{1}{2} \tag{70}$$

as expected. This supports the physical interpretation of our ansatz as representing a configuration lying 'halfway' between adjacent vacua.

4 Electroweak Baryogenesis

Now that we have identified electroweak sphaleron processes as a suitable mechanism for baryon number violation, we seek to embed these transitions within

a complete theory of baryogenesis to explain the generation of the observed mattery-antimatter asymmetry in our universe. To do so, we begin by outlining the full set of conditions that such a theory must satisfy. These features are deemed the 'Sakharov Criteria' and are given by the following:

1. Violation of baryon number (B) symmetry.
2. Violation of C (charge conjugation) and CP symmetries.
3. Departure from thermal equilibrium.

The first condition is self-evident and has already been thoroughly addressed in the preceding sections. The second condition is required because, if C and CP are exact symmetries, it is clear that the total rate to produce a baryon excess is exactly equal to the rate to produce an excess of antibaryons. (No net ΔB will be generated). Finally, the classic argument for the third condition involves considering the average of B in thermal equilibrium:

$$\begin{aligned} \langle B \rangle_T &= \mathrm{Tr}(e^{-\beta H} B) \\ &= \mathrm{Tr}[(CPT)(CPT)^{-1} e^{-\beta H} B] \\ &= \mathrm{Tr}[e^{-\beta H} (CPT)^{-1} B (CPT)] \\ &= -\mathrm{Tr}(e^{-\beta H} B), \end{aligned} \tag{71}$$

where we have used the commutation of the Hamiltonian H and the discrete operator CPT as well as the fact that B is odd under C and even under P and T. We therefore see that $\langle B \rangle_T = 0$ in equilibrium, and baryon asymmetry can only be generated by a departure from this state.

In the following section, we will see that the Electroweak sector of the Standard Model naturally satisfies all three conditions but ultimately comes up short as a complete model of baryogenesis. However, we find that with some minimal extensions, Electroweak Baryogenesis (EWBG) may yet be revived as a possible dynamical explanation of matter asymmetry.

To start, we consider the second condition of violating both C and CP within the context of the Standard Model. The question of C-violation is immediately answered by the chiral coupling of the Electroweak sector. Namely, only left-handed fermions couple to the $SU(2)$ gauge fields, while left-handed antifermions are singlets under $SU(2)$. The Electroweak theory is therefore maximally C-violating.

The subject of CP-violation is a bit more involved. The full Electroweak Lagrangian (3) contains a charged current term which may be expressed as

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} \bar{U}_L \gamma^\mu D_L W_\mu + \text{h.c.}, \tag{72}$$

where $U_L \equiv (u, c, t, ...)_L$, $D_L \equiv (d, s, b, ...)_L$. Additionally there is a quark mass term which is a matrix in the space of generations and may be diagonalized by

unitary matrices,

$$\begin{aligned} \text{diag}(m_u, m_c, m_t, ...) &= V_L^U M^U V_R^U, \\ \text{diag}(m_d, m_s, m_b, ...) &= V_L^D M^D V_R^D. \end{aligned} \tag{73}$$

In this basis of quark mass eigenstates, the charged current (72) becomes

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} \bar{U}'_L K \gamma^\mu D'_L W_\mu + \text{h.c.}, \tag{74}$$

where $U'_L \equiv V_L^U U_L$ and $D'_L \equiv V_L^D D_L$. The matrix $K \equiv V_L^U (V_L^D)^\dagger$ is the Kobayashi-Maskawa quark mass mixing matrix and contains one independent phase δ for $N = 3$ generations. Any nonzero value for this phase would translate to CP-violation, but the universe has conspired to keep $\delta \ll 1$, which is far below the threshold required for the BAU.

If we include the $SU(3)$ sector of the Standard Model, another venue for CP-violation emerges from the 'theta vacuum' structure and instanton effects. Unfortunately, measurements of the neutron electron dipole moment again bound θ_{QCD} to be too low to generate baryon asymmetry.

We now turn to the third criterion, for which we recall some terminology from thermodynamics. If a thermodynamic quantity (the 'order parameter') changes discontinuously as a function of another variable (e.g. temperature), this is deemed a 'first order phase transition.' This jump in the order parameter may decrease as another parameter of the theory is varied until, eventually, the order parameter is continuous ('continuous crossover'). If this occurs, then the point at which the transition becomes continuous is deemed a 'second order phase transition.'

Barring phase transitions, the only departure from thermal equilibrium in the cosmological history occurs from the expansion of the universe. However, as Trodden notes [15], at temperatures around the Electroweak scale, the expansion rate is well below the rate of B-violating events. Consequently, baryogenesis is impossible at these scales without the assistance of phase transitions. Fortunately, such a transition falls neatly out of the Weinberg-Salam theory from Electroweak Symmetry Breaking, where the Higgs vev $\langle\varphi\rangle = 0$ for the high temperature ground state falls to a low temperature ground state of $\langle\varphi\rangle \neq 0$.

If this Electroweak Phase Transition (EWPT) is a continuous crossover, the minimum at $\langle\varphi\rangle = 0$ becomes a local maximum at some critical temperature T_c, after which there is only one minimum at $\langle\varphi\rangle \neq 0$. The field then rolls classically to the new potential valley in a process of 'spinodal decomposition.' If the EWPT is indeed continuous or even second order, the departure from equilibrium is not strong enough to allow for successful baryogenesis. To achieve EWBG, we therefore need the EWPT to be strongly first-order (or rely on other means of escaping equilibrium like topological defects).

In a first order EWPT, the minima at $\langle\varphi\rangle = 0$ and $\langle\varphi\rangle \neq 0$ become separated by an energy barrier. At the critical temperature T_c, these minima become degenerate, while at lower T, the $\langle\varphi\rangle \neq 0$ state overtakes the other as the new global minimum. As Trodden writes, in this picture at temperatures close to

T_c, 'quantum tunneling occurs and nucleation of bubbles of the true vacuum in the sea of false begins' [15]. While these bubbles are at first too small to overcome surface tension and consequently disappear, as T_c is approached, 'critical bubbles' nucleate which are large enough to grow. As a result, these bubbles of 'true vacuum' ultimately fill the space to execute the transition. As the bubble walls pass various points in space, the Higgs vev changes rapidly (as well as other fields), sparking a dramatic departure from thermal equilibrium. Such a first order EWPT would thus provide fertile ground for a theory of baryogenesis.

We can examine the true nature of the EWPT by examining the one-loop finite temperature effective potential [15] (which I will not derive),

$$V_{\text{eff}}^{(1)}(\varphi;T) = \left(\frac{3}{32}g^2 + \frac{\lambda}{4} + \frac{m_t^2}{4v^2}\right)(T^2 - T_*^2)\varphi^2 - \frac{3g^2}{32\pi}T\varphi^3 + \frac{\lambda}{4}\varphi^4, \tag{75}$$

where $\varphi \equiv \sqrt{\phi^\dagger\phi}$ and m_t is the top quark mass. Note that the cubic term parametrizes the height of the barrier separating the two degenerate vacua at T_c. We see that increasing the Higgs mass $M_H \equiv \sqrt{2\lambda}v$ increases the strength of the quartic term relative to the cubic term. As a result, the strength of the phase transition is inversely correlated with the size of the Higgs mass. Indeed, given the measured value of $M_H \approx 125$ GeV, the Standard Model EWPT is too weak to satisfy the third Sakharov condition.

Athough the minimal Standard Model has neither enough CP-violation nor a strong enough phase transition to mediate baryogenesis on its own, a number of simple extensions have been proposed to rescue a theory of Electroweak Baryogenesis. One such approach is to view the Standard Model as an effective field theory with new physics required at some UV mass scale M. We could then add to the Lagrangian (3) additional nonrenormalizable operators, such as the dimension-six operator

$$\mathcal{O} = \frac{b}{M^2}\text{Tr}(\Phi^\dagger\Phi)\text{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{76}$$

where b is dimensionless. $\mathcal{O}$ represents the lowest-dimension CP-odd term that can be built from the Standard Model Electroweak sector. (Note: As of 2026, the bounds on the electric dipole moments of the electron and neutron are so low that such a CP-violating operator is ruled out as a mechanism for baryogenesis. However, we use $\mathcal{O}$ as a prototypical example of CP-violating extension of the Standard Model).

We now set out to examine how the higher dimensional operator $\mathcal{O}$ may be combined with sphaleron effects to give rise to a description of *local* Electroweak Baryogenesis (by 'local,' we mean that B-violating and CP-violating processes occur together near the bubble walls to generate a baryon excess). For this analysis, we assume some additional BSM component allows for a strongly first order EWPT with thin bubble walls (i.e. rapidly changing Higgs magnitude).

We first sketch a quick derivation of the effect of such an operator on anomalous baryon production from sphaleron processes. The following explanation is borrowed from Keisuke Harigaya, from whom I originally learned this material.

As we determined earlier, the rate of sphaleron events that transition between adjacent vacua due to kicks from thermal excitation is proportional to

$$\Gamma_{\text{sph}} \propto \exp\left(-\frac{E_{\text{sph}}}{T}\right), \tag{77}$$

where $E_{\text{sph}} \approx 9$ TeV. Moreover, we showed that each such transition was associated with the production of an integral baryon number ΔB proportional to the number of fermion families in the theory. However, this calculation represents the rate of events resulting in $\Delta N_{CS} > 0$ (thus generating an excess of baryons) *as well as* the rate of events with $\Delta N_{CS} < 0$ (generating an excess of *antibaryons*). As a result, sphaleron transitions do not *on their own* allow for a net baryon asymmetry.

To see how the operator $\mathcal{O}$ resolves this problem, consider its corresponding contribution to the action

$$S \supset \int d^4x \frac{b}{M^2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}). \tag{78}$$

Recalling our frequently-used definition of the baryon anomaly current (28), we may equivalently write this as

$$\begin{aligned} S &\supset \int d^4x \frac{b}{M^2} \text{Tr}(\Phi^\dagger \Phi) \partial_\mu K^\mu \\ &= \int d^4x \frac{b}{M^2} \text{Tr}(\Phi^\dagger \Phi) \partial_0 K^0 \\ &\quad + \int d^4x \frac{b}{M^2} \text{Tr}(\Phi^\dagger \Phi) \partial_i K^i \\ &= -\int d^4x \frac{b}{M^2} \partial_0 \Big(\text{Tr}(\Phi^\dagger \Phi) \Big) K^0, \end{aligned} \tag{79}$$

where in the last equality, we performed integration by parts on both terms and used the spherical symmetry of the expanding bubble to eliminate the spatial gradient term. However, we immediately identify $\int d^3x K^0$ with the Chern-Simons number from relation (29), so that we may write the action term as

$$S \supset -\frac{b}{M^2} \int dt N_{CS} \int d^3x \frac{d}{dt} \text{Tr}(\Phi^\dagger \Phi). \tag{80}$$

Furthermore, as the wall of the 'true vacuum' bubble hits a configuration at a particular point in spacetime, the value of $\text{Tr}(\Phi^\dagger \Phi)$ will change rapidly. As a result,

$$\frac{d}{dt} \text{Tr}(\Phi^\dagger \Phi) \neq 0 \tag{81}$$

during the phase transition. Consequently, this term, which is directly proportional to N_{CS}, removes the degeneracy of vacua characterized by distinct Chern-Simons numbers. This in turn leads to a *preferred* direction for sphaleron processes. In fact, noting the sign of this term and the fact that $\frac{d}{dt} \text{Tr}(\Phi^\dagger \Phi)$ will

be positive as the Higgs transitions from $\langle\varphi\rangle = 0$ to $\langle\varphi\rangle \neq 0$, we see that the operator will bias transitions with $\Delta N_{CS} > 0$, leading to a net production of baryons over antibaryons.

We now determine how this general claim manifests itself in the microscopic physics of configurations which are close to the crest between vacua as the bubble wall passes through. The following analysis originates from Shaposhnikov with our explicit steps following Dine et al. (and cited by Trodden) [10, 8, 15].

Around Electroweak temperatures, the relevant length scale of barrier crossing configurations is given by the magnetic correlation length

$$\xi \equiv (\alpha_W T)^{-1}. \tag{82}$$

We therefore divide space up into cells of volume ξ^3 and study the sphaleron dynamics of configurations in each cell as the bubble wall hits.

Consider the configuration of one such cell, which follows a trajectory in field space parametrized by τ. The energy of this configuration reaches a maximum at some τ (corresponding to the crest of the barrier), which we define as $\tau = 0$. We may construct a philosophical Lagrangian to characterize the dynamics of τ in time about the point $\tau = 0$:

$$\mathcal{L}(\tau, \dot{\tau}) = \frac{c_1}{2\xi}\dot{\tau}^2 + \frac{c_2}{2\xi^3}\tau^2 + \frac{c_3}{\xi}\frac{b}{M^2}\sigma^2\dot{\tau}, \tag{83}$$

where $\sigma^2 \equiv \mathrm{Tr}(\Phi^\dagger\Phi)$, and c_1, c_2, and c_3 are dimensionless coefficients parametrizing the possible barrier crossing pathways. The rough rationale for this Lagrangian is as follows: By construction, $\tau = 0$ is the maximum of the energy, so there can be no odd powers of τ. Additionally, as we derived, the action for the operator $\mathcal{O}$ is proportional to the time derivative of N_{CS}, so we expect $\mathcal{O}$ to give a term in the Lagrangian which is linear in $\dot{\tau}$ (τ follows a trajectory from one side of the barrier to the other).

From the Lagrangian (83), the equation of motion for τ is

$$\ddot{\tau} = \frac{c_2}{c_1\xi^2}\tau - \frac{c_3}{c_1}\frac{b}{M^2}\frac{d}{dt}\sigma^2. \tag{84}$$

In our 'thin wall' case, right after the wall strikes, the fields are not in equilibrium, and the RHS is dominated by the second term. Using an impulse approximation, we can estimate the kick that $\dot{\tau}$ receives as the bubble passes as

$$\Delta\dot{\tau} = -\frac{c_3}{c_1}\frac{b}{M^2}\Delta\sigma^2, \tag{85}$$

where $\Delta\sigma^2$ is the total amount by which σ^2 changes at the phase transition. Because this is strictly positive, $\Delta\dot{\tau}$ has a definite sign, and the velocities of configurations close to the peak are biased in the direction of excess baryon production.

To estimate the size of the resulting asymmetry, we note that, formally, the kick $\Delta\dot{\tau}$ must be larger than the initial velocity $\dot{\tau}_0$ of the configuration in order

to bias its trajectory. The fraction of configurations satisfying this condition is roughly proportional to $\Delta\dot{\tau}$, so that the net baryon number density generated may be approximated as

$$n_B \approx \Delta\dot{\tau} f \xi^{-3}, \tag{86}$$

where f is a function of the three dimensionless constants in the Lagrangian. During the Electroweak Phase Transition, the entropy density of the universe was $s \approx 45T^3$, so that the baryon to entropy ratio is obtained as

$$\frac{n_B}{s} \approx f \frac{\alpha_W^3}{45} \frac{b}{M^2} \Delta\sigma^2 \tag{87}$$

Taking the simple choice $\Delta\sigma^2 = v^2/2$ and substituting measured values for physical constants, we find the prediction

$$\frac{n_B}{s} \approx f \frac{b}{M^2} \Big(2.5 \times 10^{-2} \text{ GeV}^2 \Big). \tag{88}$$

Depending on the values of f, b, and M (and neglecting for now precision measurements of electric dipole moments), we see that it is at least feasible to achieve baryon-antibaryon asymmetry from small modifications of the Electroweak sector.

5 Conclusion

Returning to where we started, we observe that our universe exhibits a remarkable preference for matter above antimatter. Moreover, this imbalance is not just limited to our earth, or for that matter our galaxy, but rather characterizes the particle content on all scales up to the Hubble radius. We are therefore left to wonder whether the Standard Model framework we have today could possibly explain such a flagrant bias in the cosmological evolution of our universe.

The first basic question we addressed was whether this theory even had some mechanism of violating baryon number. Such an option seemed to present itself from the topological vacuum structure of the theory, with an infinite set of degenerate vacua characterized by an integral winding number N_H. Showing that transitions between such vacua gave rise to integer changes in baryon number, we naturally turned to the Electroweak instanton as a means of 'quantum tunneling' from one vacuum to another. However, we showed that the rate of such processes was exponentially suppressed by the Euclidean instanton action with a high potential barrier. Since tunneling *through* this barrier was impractical, we were motivated to consider finite temperature effects that could 'kick' configurations *over* the barrier instead. To do so, we needed to exceed a certain threshold energy of a configuration which sat at the lowest point along the ridge of finite-energy configurations separating adjacent vacua. Such a saddle-point configuration was deemed the 'sphaleron,' and it was the energy of this state which was found to govern the rate of thermal transitions between vacua.

Constructing an ansatz for such a static, classical field configuration of the Weinberg-Salam theory, we verified that this object was a saddle point of the

Electroweak energy functional with baryon number 1/2. We therefore identified Electroweak sphaleron processes as a more efficient means of baryon number violation and sought to embed this within a complete theory of Electroweak Baryogenesis.

However, reviewing the Sakharov Criteria, we found that the minimal Standard Model had neither enough CP violation nor a strong enough phase transition to dynamically generate a baryon asymmetry on its own. Practically, this amounted to the fact that the rate of sphaleron events generating a baryon excess was equal to the rate of those generating an antibaryon excess for completely degenerate N vacua. We showed, though, that this problem could be remedied by extending the Standard Model with some additional CP-violating sector such as higher-dimensional operators that could remove the degeneracy and bias sphaleron transitions toward the production of baryons. Beyond Electroweak physics, other models of baryogenesis such as Leptogenesis and Axiogenesis differ in their mechanisms of CP violation and departure from thermal equilibrium, but many of these theories similarly rely on sphaleron events to translate such physics into the dynamical production of net baryon asymmetry [9, 5]. Ultimately, then, these static, saddle point configurations represent both conceptually simple classical solutions to our equations of motion and yet profound objects whose very existence may offer us insight into the disparity between matter and antimatter in our observable universe.

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