Quantum Field Theory III Final Paper

't Hooft-Polyakov Monopoles and the Witten Effect

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Abstract

We discuss various aspects of magnetic monopoles in nonabelian gauge theories. We begin with the basics of Dirac's quantization condition, electromagnetic duality, and dyons (objects with both electric and magnetic charge). We then construct 't Hooft-Polyakov magnetic monopoles in nonabelian gauge theories and connect their existence to topology. We further show that, in the presence of a CP-symmetry breaking term in the Lagrangian, such magnetic monopoles are endowed with a fractional electric charge, which is the Witten effect. Our scope is mainly centered on the original papers of 't Hooft [1] and Witten [2].

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1 Introduction

Electric monopole charges are ubiquitous in physics. In contrast, one of Maxwell's equations:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1}$$

suggests that magnetic monopoles do not exist, which is also supported by the equation expressing the magnetic field as the curl of the vector potential (however, ambiguity in the definition of the vector potential underlies Dirac's argument for how magnetic monopoles can be constructed) [3]. The possible existence of magnetic monopoles has interesting implications in various areas of physics. One example is that, since magnetic monopoles are very massive (as we shall show in this work) [4, 5] they could not be produced in accelerators or the present universe. Instead, any monopoles around today should date from the early universe, and estimates suggest we should be able to observe them. However, we do not, constituting a problem in cosmology [5].

In the 1930s, Dirac showed a quantization condition for the product of electric charge and magnetic charge, which is indeed consistent with the quantized electric charge we observe in the universe. This theory argued that the existence of monopoles was consistent with the observed world. However, there was not an argument that monopoles were necessary [4]. In the 1970s, 't Hooft and Polyakov showed that these are inevitable constructions in common nonabelian gauge theories [1, 4]. The role played by spontaneous symmetry breaking and the resulting topological windings is particularly significant, as we shall explore in detail.

Another interesting avenue is motivated by considering the existence of dyons, which are objects that contain both an electric charge and a magnetic charge. There is a quantization condition between two dyons, analogous to Dirac's quantization condition, which can be seen based on symmetry under duality transformations, which exchange electric and magnetic quantities [4]. Consistent with this quantization condition, it was shown by Witten [2] that if a CP symmetry breaking term is included in the Lagrangian, then monopoles are endowed with fractional electric charge.

We shall discuss these findings in detail in the coming sections. In the rest of this section, we shall discuss Dirac's quantization condition, duality, and dyons. In Section 2 we shall discuss the 't Hooft-Polyakov monopoles. We shall discuss the specifics in a model nonabelian gauge theory with spontaneous symmetry breaking, computing the topological charge of magnetic monopoles, and also their mass. We shall also give a general topological argument pertaining to more general gauge groups undergoing spontaneous symmetry breaking. In Section 3 we shall discuss the Witten effect by which monopoles are endowed with fractional electric charge. We shall conclude in Section 4.

1.1 Dirac's Quantization Condition

Let us consider the quantum mechanics of a charged particle in the vicinity of a magnetic monopole at the origin. The dynamics are described by a Schrodinger equation:

$$i\partial_t\psi(t,\mathbf{r}) = -\frac{1}{2m}(\boldsymbol{\nabla} - ie\mathbf{A}(\mathbf{r}))^2\psi(t,\mathbf{r}) + V(\mathbf{r})\psi(t,\mathbf{r}) \tag{2}$$

When observed from the outside (specifically for a radius $r > r_0$, such that we are not interested for the time being in what happens inside the radius r_0), the magnetic monopole is defined by its magnetic field which is analogous to that of an electric monopole:

$$\mathbf{B}(\mathbf{r}) = \frac{g}{4\pi r^2}\hat{\mathbf{r}} \tag{3}$$

We would like to define this field via a vector potential $\mathbf{A}$. However, since the magnetic field is the curl of this vector potential, at a first glance it would seem there is no way it can have a divergence as the field above clearly does. However, we can exploit the gauge ambiguity in the definition of the vector potential, making different definitions for different angular regions of the sphere and stitching them together via gauge transformations. That is, the two definitions of the vector potential we use must have the same curl where they overlap. Consider, for instance:

$$\mathbf{A}_N(\mathbf{r}) = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\varphi}, \quad \mathbf{A}_S(\mathbf{r}) = -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\hat{\varphi} \tag{4}$$

and these fields are well-defined everywhere except for the south pole $\theta = \pi$ for $\mathbf{A}_N$ and the north pole $\theta = 0$ for $\mathbf{A}_S$ (the angle where they are not defined is referred to as a Dirac string which, intuitively, is an infinitesimally thin solenoid carrying the necessary flux to make the divergence of $\mathbf{A}$ zero, as shown in Fig. 1). The curl of both of these fields indeed gives $\mathbf{B}$:

$$\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi(\mathbf{r}))\hat{\mathbf{r}} = \pm\frac{g}{4\pi r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(1\mp\cos\theta)\hat{\mathbf{r}} = \frac{g}{4\pi r^2}\hat{\mathbf{r}} \tag{5}$$

In their region of overlap, these gauge potentials must differ by a gauge transformation and thus by a gradient:

$$\mathbf{A}_N - \mathbf{A}_S = \frac{g}{4\pi r}\frac{2}{\sin\theta}\hat{\varphi} = -\boldsymbol{\nabla}\chi(\mathbf{r}) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\chi(\mathbf{r})\hat{\varphi} \tag{6}$$

$$\Rightarrow \quad \chi(\mathbf{r}) = -\frac{g}{2\pi}\varphi \tag{7}$$

This is not a single-valued function. However, since the gauge transformation acts only through $e^{-ie\chi(\mathbf{r})}$ as:

$$\psi(\mathbf{r}) \to e^{-ie\chi(\mathbf{r})}\psi(\mathbf{r}), \quad \mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla}\chi(\mathbf{r}) = \mathbf{A} - \frac{i}{e}e^{ie\chi(\mathbf{r})}\boldsymbol{\nabla}e^{-ie\chi(\mathbf{r})} \tag{8}$$

it only matters that $e^{-ie\chi(\mathbf{r})}$ be single-valued. This requires that $e^{-ie\chi(\varphi=0)} = e^{-ie\chi(\varphi=2\pi)}$. Thus we arrive at Dirac's quantization condition:

$$\frac{eg}{2\pi} = n \in \mathbb{Z} \tag{9}$$

This says that if there is even a single magnetic monopole of charge g in the universe, electric charge must be quantized [4, 6]. However, we also understand the quantization of electric charge as a consequence of the compact structure of the U(1) gauge group, and such a gauge group exists in "unified" gauge theories [5]. The 't Hooft-Polyakov monopole construction, which we shall explain in detail in the next section, can be understood as a sort of converse to Dirac's argument [5]. Dirac argued that if magnetic monopoles exist, electric charge is quantized, and 't Hooft-Polyakov monopoles imply that in "unified" gauge theories (where electric charge is quantized) there must exist magnetic monopoles.

1.2 Duality and Dyons

Maxwell's Equations can be written in block matrix form with both electric and magnetic charges and currents:

$$\begin{pmatrix} \boldsymbol{\nabla}\cdot & 0 \\ 0 & \boldsymbol{\nabla}\cdot \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\nabla}\times & 0 \\ 0 & \boldsymbol{\nabla}\times \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \partial_t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix} \tag{10}$$

Consider the below class of transformations parametrized by an angle α, referred to as duality transformations:

$$\begin{gathered} \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \quad \begin{pmatrix} \rho_e' \\ \rho_m' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}, \\ \begin{pmatrix} \mathbf{J}_e' \\ \mathbf{J}_m' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix} \end{gathered} \tag{11}$$

It is easy to see that Maxwell's Equations (Eqs. 10) are invariant under these transformations, since the only matrices in Eqs. 10 are the identity matrix and the antisymmetric matrix, which are both invariant under rotations.

Motivated by the duality symmetry, if electric and magnetic monopoles both exist, we must also accept the existence of dyons, which are compound objects with both electric and magnetic charge. Consider the scenario where we argued the existence of the Dirac quantization condition based on the presence of a single electric monopole e and single magnetic monopole g. Under a duality transformation, the electric monopole becomes the dyon $(e_1, g_1) \equiv (e\cos\alpha, -e\sin\alpha)$ and the magnetic monopole becomes the dyon $(e_2, g_2) \equiv (g\sin\alpha, g\cos\alpha)$. The quantity:

$$e_1 g_2 - e_2 g_1 = eg = 2\pi n \tag{12}$$

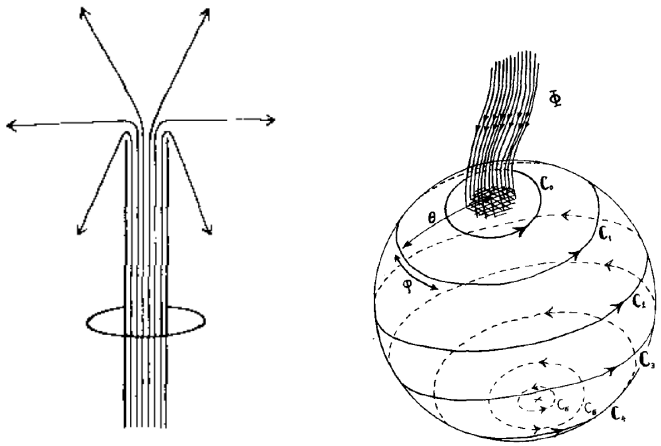

Figure 1: Left: Dirac string configuration (figure from [5]): the flux leaving and entering the "monopole" is the same. Right: 't Hooft configuration (figure from [1]): there is net flux through the sphere, and various contours are shown.

is thus also quantized. This equation is solved, for example, by [4]:

$$e_i = n_i e + m_i f \tag{13a}$$

$$g_i = \frac{2\pi}{e} m_i \tag{13b}$$

where e, f are arbitrary real numbers. This solution says that each quanta of magnetic charge can be accompanied by a fractional electric charge [4]. Thus it is very possible that magnetic monopoles carry such electric charge, and we shall see there are situations where this is indeed the case, when we discuss the Witten Effect.

2 't Hooft-Polyakov Monopoles

The previous section considered the monopole as a point object, meaning we only were interested in what happens outside the monopole. Let us now attempt to investigate its interior and how it can be constructed. To do so, we shall investigate 't Hooft-Polyakov monopoles which arise in gauge theories with spontaneous symmetry breaking.

2.1 't Hooft's Original Motivation

A monopole is, essentially, a point of nonzero divergence of the magnetic field. It provides a way for tubes of flux, such as those in superconductors, to begin and end [1]. Thus the question of whether flux tubes can begin or end in various gauge theories is related to the existence of monopoles in these theories, and was part of 't Hooft's original motivation.

Roughly speaking, in regions where there is no flux, the gauge potential will be locally related to the vacuum gauge potential $\mathbf{A} = 0$ by a gauge transformation. We

can associate this smoothly varying $\mathbf{A}$ with the smoothly varying gauge transformations that take $\mathbf{A}$ to be the vacuum. Around the flux tube, the gauge potential must take the form of the usual electromagnetic gauge potential and thus we see that a loop around the flux tube has a winding number related to the flux in the tube. Associating the gauge potential at each point with a gauge transformation $\in$ U(1) we have a mapping from $S^1 \to S^1$ which is characterized by the topological invariants in the group $\pi_1(S^1) = \mathbb{Z}$. Thus there is no way to smoothly deform the gauge potential into the vacuum gauge potential $\mathbf{A} = 0$ by remaining in the Abelian group U(1) since the winding numbers are different. This implies that the flux tubes cannot begin and end anywhere, since an endpoint would mean that there is a continuous way to go from the winding gauge potential about the flux tube to the non-winding gauge potential in vacuum. However, if the gauge group is larger, say SU(2), then we are not confined to mappings from a loop onto S^1 but instead S^3. Thus the relevant topological invariant is $\pi_1(S^3) = 0$. That is, there is no conserved topological quantity. Thus the flux tube is allowed to have a beginning or end. Thus monopoles could exist in such a theory.

Also note that while the flux tube may at first glance pictorially look like the Dirac string, it is very different. In the case of the Dirac string, there is uniformly outgoing flux everywhere on a sphere enclosing the monopole, except for a singular point on the sphere where all the flux enters the sphere for a net flux of zero. However, in the case we are currently considering, the only place a net nonzero flux enters or leaves is through the tube and nowhere else. See Fig. 1 for a comparison of the two situations.

2.2 Energy Density and Explicit Construction in Spontaneous Symmetry Broken Nonabelian Gauge Theory

Let us consider a gauge field coupled to a scalar matter field that lives in a quartic potential [1, 6]:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\phi^a D_\mu\phi^a - \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2 \tag{14}$$

$$\text{where} \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon^{abc}A^b_\mu A^c_\nu, \quad D_\mu\phi^a = \partial_\mu\phi^a + e\varepsilon^{abc}A^b_\mu\phi^c \tag{15}$$

where $a, b, c = 1, 2, 3$ label the adjoint representation of SU(2), and the structure factor $f^{abc} = \varepsilon^{abc}$. Due to spontaneous symmetry breaking, the expectation value of ϕ^a takes a fixed direction. Let us momentarily define the basis such that $\phi^a = v\delta^{a3}$. Since $\phi = \phi^a T^a$ where T^a are the group generators, we ask which symmetries of the form $U\phi U = \phi$ remain for the expectation value vT^3. We see that if $U = T^3$ this can be satisfied and otherwise not. Thus only the T^3 symmetry survives, and the symmetry group SU(2) is broken down to U(1) [7]. The two massless Goldstone bosons of the theory corresponding to T_1 and T_2 are "eaten" by two components of the gauge field, giving those components mass and a longitudinal component (we can call these components of the gauge field W bosons). The remaining component of the gauge field, associated with the unbroken symmetry T_3 is massless (we can call this a photon). The remaining component of the scalar field that was not eaten by the gauge bosons is the Higgs particle [1, 7].

With the spontaneous symmetry breaking, we write $\phi^a = v\delta^{a3} + \Delta\phi^a$. Then:

$$\mathcal{L} \supset \frac{1}{2} D^\mu \phi^a D_\mu \phi^a \supset \frac{1}{2} e^2 \varepsilon^{abc} \varepsilon^{ade} A^b_\mu \phi^c A^{d\mu} \phi^e \supset \frac{1}{2} v^2 e^2 \varepsilon^{abc} \varepsilon^{ade} A^b_\mu \delta^{c3} A^{d\mu} \delta^{e3} \tag{16}$$

$$= \frac{1}{2} v^2 e^2 (\delta^{bd}\delta^{ce} - \delta^{be}\delta^{dc}) A^b_\mu \delta^{c3} A^{d\mu} \delta^{e3} = \frac{1}{2} v^2 e^2 A^b_\mu A^{b\mu} - \frac{1}{2} v^2 e^2 A^3_\mu A^{3\mu} \tag{17}$$

$$= \frac{1}{2} v^2 e^2 (A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}) \tag{18}$$

Thus the mass of the massive gauge bosons is:

$$M_W = ev \tag{19}$$

We shall use this result later when discussing the mass of the monopole.

The energy density for this theory (i.e. the 00 component of the energy-momentum tensor) is [6]:

$$\Theta_{00} = \frac{1}{2}(\mathbf{E}^a \cdot \mathbf{E}^a + \mathbf{B}^a \cdot \mathbf{B}^a + D_0\phi^a D_0\phi^a + D_i\phi^a D_i\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2, \tag{20}$$

$$E^{ai} = -F^{a0i}, \quad B^{ai} = -\frac{1}{2}\varepsilon^{ijk} F^a_{jk} \tag{21}$$

This form of energy density looks very familiar based on our experience from electromagnetism. The first two terms look like the usual energy density in free space $\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$: the only difference now is that there are multiple gauge fields. Then there is a sum over all derivatives of the fields ϕ^a, which also makes sense. And finally, we have the potential energy term. This energy density is positive semidefinite, $\Theta_{00} \geq 0$, since all terms are "magnitudes" of something. When is the energy density 0? Each term can be separately minimized: namely, we see that $F^{a\mu\nu} = 0$, $D^\mu\phi^a = 0$, $V(\phi) = 0$. This occurs when the scalar field has a constant vacuum expectation value at all points in space:

$$\langle \phi^a \phi^a \rangle = v^2 \tag{22}$$

This is the spontaneous symmetry breaking, where the SU(2) group is symmetry-broken down to the U(1) group. The vacuum can thus be defined as the set of all configuations of the scalar field which minimize the energy density. These configurations $\langle \phi^a \phi^a \rangle = v^2$ define a two-sphere S^2_V (label V is for vacuum), since $a = 1, 2, 3$. Note that in what follows we use the same notation for the field and its expectation value, as also done in [6].

To have finite energy excitations, it is easy to see that they must be confined to some region in space, and far from that region we must recover the vacuum condition $\langle \phi^a \phi^a \rangle = v^2$. Thus, at each angular point at infinity, we have some configuration of fields for which $\langle \phi^a \phi^a \rangle = v^2$, specified by a point on the two-sphere S^2_V. This is a mapping between two two-spheres. Such a mapping is characterized by a homotopy group, which colloquially describes how many times one of the spheres winds around the other. For the case of a pair of two spheres wrapping around each other, the homotopy group is the integers: $\pi_2(S^2) = \mathbb{Z}$. This concept is illustrated in Fig. 2.

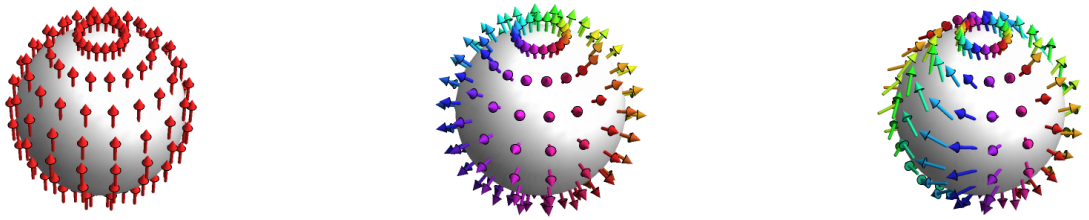

Figure 2: Illustration of different maps $S^2_A \to S^2_B$ corresponding to different elements of the second homotopy group $\pi_2(S^2)$. Left: 0 winding, all points on the sphere S^2_A map to the same point on S^2_B. Middle: 1 winding, each point on the sphere S^2_A maps to a different point on S^2_B. Right: 2 windings, each point on S^2_B is mapped to from two points on S^2_A. The color coding in each of the figures corresponds to the azimuthal angle of the mappings, which have the functional form $(\sin\theta\cos(n\varphi), \sin\theta\sin(n\varphi), \cos\theta)$, where $n \in \mathbb{Z}$ dictates how many windings there are, which is 0,1,2 respectively for the three figures above. Thus the colors also wind 0,1,2 times around the azimuthal direction.

Let us now consider the field configurations that have nontrivial winding number. The absence of gauge fields would imply that the total energy is:

$$\int d^3x \Theta_{00} = \int d^3\left(\frac{1}{2}\boldsymbol{\nabla}\phi^a\boldsymbol{\nabla}\phi^a + \frac{1}{2}\partial_t\phi^a\partial_t\phi^a + V(\phi^a)\right) \geq \int d^3\frac{1}{2}\boldsymbol{\nabla}\phi^a\boldsymbol{\nabla}\phi^a \tag{23}$$

$$= \int d^3x\frac{1}{2}((\partial_r\phi^a)^2 + (\hat{\mathbf{r}}\times\boldsymbol{\nabla}\phi^a)^2) \geq \int d^3x\frac{1}{2}(\hat{\mathbf{r}}\times\boldsymbol{\nabla}\phi^a)^2 \tag{24}$$

The angular part of the magnitude of the gradient that we isolated above must be nonzero if there is to be a nontrivial winding number about the sphere. Intuitively, one can expect the integrand to scale as $1/r^2$ and so the integral of it goes as [6]:

$$\int d^3x\Theta_{00} \geq \int r^2 dr d\Omega\frac{1}{r^2} \tag{25}$$

which diverges linearly. Thus we cannot have finite energy configurations with nontrivial topological winding without gauge fields. The role of gauge fields will be to cancel the ∂_μ term when we consider the covariant derivative D_μ in the energy density.

To see this, consider that the energy density and thus the square of the covariant derivatives must fall off faster than $1/r^2$ as we go to infinity, and thus the covariant derivatives must fall off faster than $1/r$. We want to satisfy, for $\phi^a = vn^a$ such that $n^a n^a = 1$:

$$D_\mu n^a = \partial_\mu n^a + e\varepsilon^{abc}A^b_\mu n^c = 0 \tag{26}$$

to order $1/r$. We can treat, for a given μ, the a index like a vector index and decompose A^a_μ into its parts parallel and perpendicular to n^a:

$$A^a_\mu = (A^b_i n^b)n^a + \mathcal{A}^a_i \tag{27}$$

where $\mathcal{A}_i^a$ is the perpendicular part. Then:

$$\partial_\mu n^a + e\varepsilon^{abc} A_\mu^b n^c = 0 \tag{28}$$

$$\varepsilon^{ade} n^e \partial_\mu n^a + e\varepsilon^{ade} n^e \varepsilon^{abc} A_\mu^b n^c = 0 \tag{29}$$

$$\varepsilon^{ade} n^e \partial_\mu n^a + e n^e (\delta^{db}\delta^{ce} - \delta^{dc}\delta^{eb}) A_\mu^b n^c = 0 \tag{30}$$

$$\varepsilon^{ade} n^e \partial_\mu n^a + e n^e A_\mu^d n^e - e n^e A_\mu^e n^d = 0 \tag{31}$$

$$\varepsilon^{ade} n^e \partial_\mu n^a + e(A_\mu^d - (n^e A_\mu^e) n^d) = 0 \tag{32}$$

$$\varepsilon^{ade} n^e \partial_\mu n^a + e\mathcal{A}_\mu^d = 0 \tag{33}$$

Thus we see that the perpendicular component of A_μ^a is fixed. However, the parallel part is free, as we can also see explicitly by swapping dummy indices and using the antisymmetric nature of the Levi-Civita tensor:

$$e\varepsilon^{abc}(A_\mu^d n^d) n^b n^c = e\varepsilon^{acb}(A_\mu^d n^d) n^c n^b = -e\varepsilon^{abc}(A_\mu^d n^d) n^c n^b = 0 \tag{34}$$

Thus, in general:

$$A_\mu^a = (A_i^b n^b) n^a - \frac{1}{e}\varepsilon^{abc} n^b \partial_\mu n^c \tag{35}$$

$$A_\mu^a = \frac{1}{v}\tilde{A}_\mu \phi^a - \frac{1}{ev^2}\varepsilon^{abc}\phi^b \partial_\mu \phi^c \tag{36}$$

where $\tilde{A}_\mu \equiv A_i^b n^b$ is arbitrary. With some algebra, it can be shown (Appendix A) that the corresponding $F^{a\mu\nu}$ is:

$$F^{a\mu\nu} = \frac{1}{v}\phi^a \left(\partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu - \frac{1}{ev^3}\varepsilon^{efg}\phi^e (\partial^\mu \phi^f)(\partial^\nu \phi^g) \right) \tag{37}$$

We see that the gauge field is thus along ϕ^a which is the U(1) direction. Thus this is the usual electromagnetic field. We in fact have the magnetic field as:

$$B^i = -\frac{1}{2}\varepsilon^{ijk} \left(\partial^j \tilde{A}^k - \partial^k \tilde{A}^j - \frac{1}{ev^3}\varepsilon^{efg}\phi^e (\partial^j \phi^f)(\partial^k \phi^g) \right) \tag{38}$$

and the magnetic charge as:

$$g = \int_S B^i dS^i = \int_S -\frac{1}{2}\varepsilon^{ijk}\left(\partial^j \tilde{A}^k - \partial^k \tilde{A}^j\right) dS^i + \int_S -\frac{1}{2ev^3}\varepsilon^{ijk}\varepsilon^{efg}\phi^e (\partial^j \phi^f)(\partial^k \phi^g) dS^i \tag{39}$$

The first term vanishes by Stokes' theorem, since the curl of a vector field on a surface is the line integral of the vector field on the boundary but if the surface is closed there is no boundary and the integral is 0. The second term can be evaluated by using a well-known result from topology that [3]:

$$N = \frac{1}{8\pi}\int_S dS_i \varepsilon^{ijk}\varepsilon^{efg}\tilde{\phi}^e (\partial^j \tilde{\phi}^f)(\partial^k \tilde{\phi}^g) \tag{40}$$

is the winding number $\pi_2(S^2)$ of the map between two two-spheres, and $\tilde{\phi}^a = \phi^a/v$. Then:

$$g = \frac{4\pi N}{e} \tag{41}$$

We have thus found that, for finite energy excitations that are topologically nontrivial, it is necessary to have gauge fields that behave like magnetic monopoles to ensure the energy density is indeed finite. We must also compare our finding to the previous $g = 2\pi n/e$ obtained by the Dirac argument. The factor of 2 difference is due to the fact that the eigenvalues of T^3 (the Pauli-z matrix times 1/2) are $\pm 1/2$ and so the electric charge here can be $e/2$, and plugging this in we find consistency [6].

One should also note that the finite energy excitations, since they are topologically distinct from the vacuum state, are local minima of the energy in that they cannot be continuously deformed to the vacuum state while keeping finite energy. Thus they are stable, since they are local minima, and can be viewed as "soliton" solutions [5].

2.3 Hedgehog Configuration and Monopole Mass

Let us compute explicitly the quantities in the previous section for a particular "hedgehog" configuration, and also the mass of this monopole. This configuration is described by:

$$\phi^a(t, \mathbf{r}) = v\delta^{ai} n^i = v n^a \tag{42}$$

where n^i is the spatial unit vector. The reason for the name hedgehog is thus evident, since associating the $a = 1, 2, 3$ indices with spatial indices $i = 1, 2, 3$ the field ϕ^a points radially outwards. For the magnitude, we have used that the ϕ^a will have magnitude v due to the spontaneous symmetry breaking. The corresponding value of A^a_μ is (using Eq. 36):

$$A^a_i = -\frac{1}{e}\varepsilon^{abc} n^b \partial_i n^c = -\frac{1}{e}\varepsilon^{abc} n^b \left(\frac{\delta^{ic}}{r} - \frac{r^c r^i}{r^3}\right) = -\frac{1}{e}\left(\frac{\varepsilon^{abi} n^b}{r} - \frac{\varepsilon^{abc} n^b n^c n^i}{r}\right) = -\frac{1}{er}\varepsilon^{abi} n^b \tag{43}$$

where we have set $\tilde{A}_\mu = 0$ since it is arbitrary and will not show up in the magnetic charge expression either, which also makes $A^a_0 = 0$ which is why we have only spatial components above. In the last step, we have used that $\varepsilon^{abc} n^b n^c = 0$. This configuration is shown in Fig. 3. The field is then (using Eq. 37):

$$\begin{aligned} F^{ajk} &= n^a\left(\varepsilon^{efg} n^e (\partial^j n^f)(\partial^k n^g)\right) = -\frac{1}{e} n^a \varepsilon^{efg} n^e \left(\frac{\delta^{jf}}{r} - \frac{n^f n^j}{r}\right)\left(\frac{\delta^{kg}}{r} - \frac{n^g n^k}{r}\right) \\ &= -\frac{1}{er^2} n^a \varepsilon^{ejk} n^e \end{aligned} \tag{44}$$

and:

$$B^i = -\frac{1}{2}\varepsilon^{ijk}\left(-\frac{1}{er^2}\varepsilon^{ejk} n^e\right) = \frac{1}{2er^2} 2\delta^{ie} n^e = \frac{1}{er^2} n^i \tag{45}$$

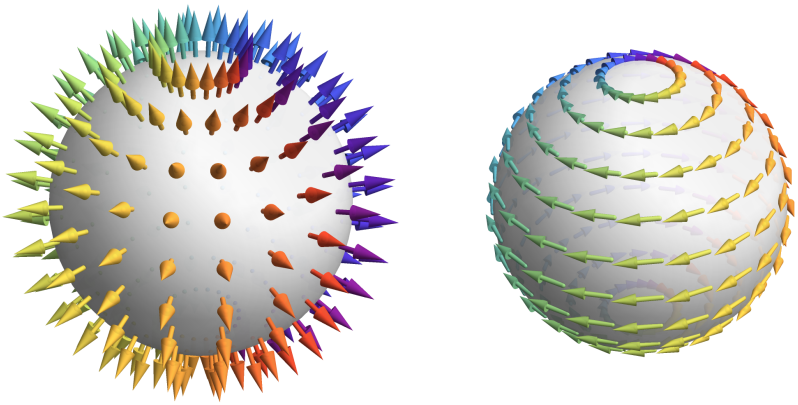

Figure 3: Hedgehog field configurations. Left: plot of $\phi^a = vn^a$, where the a degree of freedom has been associated with the spatial coordinates to clarify the hedgehog nomenclature. Right: plot of $A_i^a = -\frac{1}{er}\varepsilon^{abi}n^b$ for $a = 3$ where now the vectors are actual spatial vectors. The configuration is effectively $-\hat{\mathbf{z}} \times \hat{\mathbf{r}}$ for A_i^3. For A_i^1 or A_i^2 one would replace $\hat{\mathbf{z}}$ with $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ respectively and thus obtain rotated configurations of the same plot. In both plots, colors correspond to azimuthal direction to make the winding explicit.

where in the second equality we have used the identity $\varepsilon^{ijk}\varepsilon^{ejk} = 2\delta^{ie}$. This magnetic field points radially outwards, as we would expect. The magnetic charge can be directly computed:

$$g = \int_S B^i dS^i = (4\pi r^2)\frac{1}{er^2} = \frac{4\pi}{e} \tag{46}$$

consistent with the $N = 1$ solution in the previous section.

Let us now estimate the mass of this configuration via a lower bound. The massive gauge fields, having mass M_W, thus decay away from the origin as $e^{-M_W r}$. Therefore we expect our above solution is approximately valid for $r \geq 1/M_W$. Thus the energy density Eq. 20 of the configuration, which is the mass for a static configuration, can be approximately lower bounded [4]:

$$\Theta_{00} \geq 4\pi \int_{1/M_W}^{\infty} r^2 dr \frac{1}{2}(B^i B^i) = \frac{2\pi}{e^2}\int_{1/M_W}^{\infty} dr \frac{1}{r^2} = \frac{2\pi M_W}{e^2} = \frac{2\pi v}{e} \tag{47}$$

Thus we see that the monopole mass is related to the mass of the gauge bosons.

To be more precise in computing the mass, we can set up the full energy density for a spherically symmetric solution (corresponding to the same magnetic charge $N = 1$ as before) with arbitrary radial dependence [1]. We define, more generally:

$$\phi(\mathbf{r}) = n^a v\rho f(\rho), \quad A_i^a = \varepsilon^{abi}n^b v\rho a(\rho), \quad \rho = ver, \quad \beta = \frac{\lambda}{e^2} \tag{48}$$

In terms of such fields, we have (since the monopole is stationary and at rest, the energy

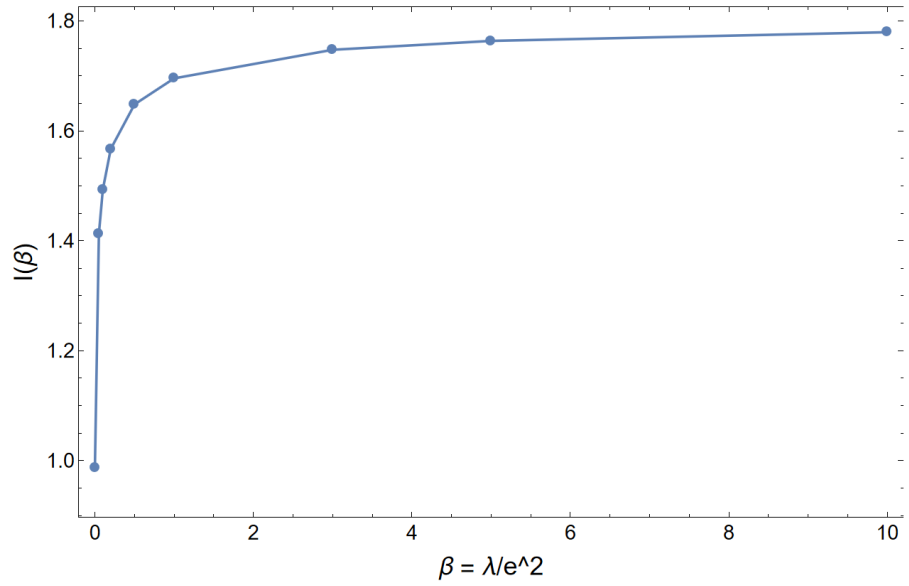

Figure 4: Value of $I(\beta)$ obtained numerically. The line is meant as a guide to the eye.

is equal to the Lagrangian which is equal to the mass) [1]:

$$M_M = -L = \frac{4\pi M_W}{e^2}\int_0^\infty \rho^2 d\rho (6a^2(\rho) + 2\rho^2 a^3(\rho) + \frac{\rho^4}{2}a^4(\rho) + 4\rho a(\rho)\frac{\partial}{\partial\rho}a(\rho) + \rho^2\left(\frac{\partial}{\partial\rho}a(\rho)\right)^2 + \frac{3}{2}f(\rho)^2 + 2\rho^2 a(\rho)f(\rho)^2 + \rho^4 a(\rho)^2 f(\rho)^2 + \rho f(\rho)\frac{\partial}{\partial\rho}f(\rho) + \frac{1}{2}\rho^2\left(\frac{\partial}{\partial\rho}f(\rho)\right)^2 + \frac{\beta}{4}\rho^4 f(\rho)^4 - \frac{\beta}{2}\rho^2 f(\rho)^2 + \frac{\beta}{4}) \tag{49}$$

as derived in Appendix B. This expression is of the form:

$$M_M = \frac{4\pi M_W}{e^2} I(\beta), \quad 1 \lesssim I(\beta) \lesssim 2 \tag{50}$$

A plot of $I(\beta)$ obtained numerically via trial function minimization in Mathematica is given in Fig.4. The range of $I(\beta)$ is consistent with the numbers mentioned in Refs. [1, 5]. Thus our original lower bound $M_M \approx 2\pi M_W/e^2$ was too small perhaps by a factor of 2-4, which is quite expected. Thus we have shown that, replacing $e^2/4\pi$ by the fine structure constant [1]:

$$M_M \approx 137\, I(\beta) \tag{51}$$

We see that monopoles are much heavier than the gauge bosons of the theory. This makes it difficult for monopoles to be created in accelerators or supernovae, however they may have been created in the early universe [4].

2.4 General Topological Arguments

In the previous sections, we made explicit computations for a particular gauge theory. In this section, let us make more general topological arguments.

Consider a gauge theory for a gauge group G which undergoes spontaneous symmetry breaking down to a subgroup of G called H. There is some potential $U(\phi)$ which has many degenerate minima. Let us denote ϕ_0 as one of these minima, which is left invariant under the group H. The value ϕ_0 is then transformed to the other minima via transformations in the coset space G/H. In the example of the previous section, G was SU(2) and H was U(1). The fields on the sphere at infinity have a symmetry under H, and so the field on the sphere at infinity is really a mapping from the two-sphere to the coset space G/H. The topological character of this winding number is given by the second homotopy group $\pi_2(G/H)$. The value of this winding number is a topological invariant, and is conserved in dynamics since it cannot change under continuous deformations [4, 5].

We shall next connect this topological invariant of winding numbers of mappings on spheres to a topological invariant of winding numbers of mappings on loops, which shall connect to our original Dirac quantization argument which involved loops rather than surfaces. This shall further illuminate how we obtained the same quantization condition via the 't Hooft-Polyakov construction of the monopole as in the Dirac case.

Let us denote the mapping from $\pi_2(G/H)$ to S_2 as $\phi(\theta, \varphi)$ where θ, φ are the polar and azimuthal angles respectively. Consider the two hemispheres of the sphere, for which the polar angle $0 \leq \theta \leq \pi/2$ and $\pi/2 \leq \theta \leq \pi$. On each hemisphere there exists a smooth gauge transformation that takes $\phi(\theta, \varphi)$ to the value ϕ_0. Define these transformations:

$$\Omega_N(\theta, \varphi)\phi(\theta, \varphi) = \phi_0, \quad 0 \leq \theta \leq \pi/2 \tag{52}$$

$$\Omega_S(\theta, \varphi)\phi(\theta, \varphi) = \phi_0, \quad \pi/2 \leq \theta \leq \pi \tag{53}$$

On the equator $\theta = \pi/2$, we can define the operation $\Omega_N(\pi/2, \varphi)\Omega_S^{-1}(\pi/2, \varphi)$. This takes $\Omega_N(\pi/2, \varphi)\Omega_S^{-1}(\pi/2, \varphi)\phi_0 = \phi_0$. Thus $\Omega_N(\pi/2, \varphi)\Omega_S^{-1}(\pi/2, \varphi) \in H$. Thus, we have a loop in H that is traversed by the parameter φ. This procedure allows us to take any mapping from S^2 to G/H and associate with it a mapping from the one-sphere (circle) S^1 to H. In fact, it can be shown that the groups $\pi_2(G/H)$ and $\pi_1(H)$ are isomorphic to one another if G is simply connected [5]. Thus the topology of the mappings on surfaces and the mappings on loops are related.

Let us connect this to the physics we have discussed. In our previous example, we had that G was SU(2) and H was U(1). Note that SU(2) is indeed simply connected. The coset is SU(2)/U(1), which is the subgroup generated by $T_1 = \sigma_x/2$ and $T_2 = \sigma_y/2$, since the single generator in U(1) is $T_3 = \sigma_z/2$. Topologically, this coset is a two-sphere S^2, as we discovered earlier based on the fact that the various ϕ^a for which $\phi^a\phi^a = v^2$ live on a two-sphere. It is a well known result in topology that $\pi_2(S^2) = \mathbb{Z}$. Further, since $U(1)$ is topologically like S^1, we examine $\pi_1(S^1)$ which is also well known to be $\mathbb{Z}$. Thus indeed we have $\pi_2(G/H) = \pi_1(H)$ in this example, and the monopole charges can take any value in the integers.

Thus we can argue the necessity of magnetic monopoles in such gauge theories as follows. There do exist stable (due to their topological character) and finite energy (since the derivatives that arise from nontrivial winding are canceled by the gauge fields to yield covariant derivatives that decay faster than $1/r$) excitations of the gauge theory. The topological winding number of these solutions is the magnetic monopole charge, viewed either from the perspective of $\pi_2(G/H)$ or $\pi_1(H)$ [5].

Moreover, we remark that we can have fields with nontrivial winding on the sphere at infinity without any singularity of the field. This comes about because, as we move into the center of the sphere, there we are no longer required to have the fields aligned with the energy minimum in question, and thus instead of loops in H we have full access to G, which means that loops that were not contractible to each other in H are contractible to each other in G, without needing any singularity [4]. This can be stated more formally. Every path in H is a path in G. Then, we see that there is a mapping $\pi_1(H) \to \pi_1(G)$. This map has a kernel, the set of values in $\pi_1(H)$ that are taken to the identity in $\pi_1(G)$. In words, this tells us which loops that wind around H are trivial in G. If these are only trivial loops, then we do not have monopoles without singularity. However, if there are nontrivial loops in H that are trivial in G, then we can have monopoles without singularity. This condition turns out to be both necessary and sufficient [4].

3 Witten Effect

The Witten Effect is the effect by which magnetic monopoles acquire fractional electric charge. Specifically, when we add a P and CP-symmetry breaking term parametrized by θ to the Lagrangian:

$$\Delta\mathcal{L} = -\frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F^a_{\mu\nu}F^a_{\sigma\rho} \tag{54}$$

it turns out that magnetic monopoles acquire fractional electric charge. We showed (Eq. 13) that there was nothing wrong with this from the Dirac quantization point of view, and also argued that this is also related to duality. Indeed, a nonzero θ can arise from a duality transformation. Consider a duality transformation of the usual electromagnetic Lagrangian (upto coefficients) as in Eq. 11, by an angle α:

$$\begin{aligned}(\mathbf{E}^2 - \mathbf{B}^2) \to ((\mathbf{E}\cos\alpha + \mathbf{B}\sin\alpha)^2 - (\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha)^2) \\ = \cos 2\alpha(\mathbf{E}^2 - \mathbf{B}^2) - 2\sin 2\alpha(\mathbf{E}\cdot\mathbf{B})\end{aligned} \tag{55}$$

We see that nonzero α generates a $\mathbf{E}\cdot\mathbf{B}$ term which is well-known to be the Abelian version of $\Delta\mathcal{L}$. Duality can still be considered a symmetry of the system since the $\Delta\mathcal{L}$ term is a total derivative (we shall derive and discuss this later in this section, where we see that Maxwell's Equations only depend on the derivatives of θ) and thus does not affect the equations of motion in the bulk, only at surfaces. Thus the idea that magnetic monopoles have fractional electric charge in a universe with nonzero θ is not too hard to digest, from the viewpoint of duality transformations as well.

It is significant that the above $\Delta\mathcal{L}$ breaks CP symmetry. A CP transformation takes $(e_i, g_i) \to (-e_i, g_i)$. Let us assume, motivated by the Dirac quantization condition that $g_i = 2\pi/e$ and that e_i is the electric charge. Since both the original particle and its CP transformed version must be valid particles in the theory, we have the below quantization condition between them:

$$(e_1 g_1 - (-e_1)g_1) = 2\pi n \Rightarrow 2e_1 = en \tag{56}$$

Thus:

$$e_1 = \frac{e}{2}n \tag{57}$$

Moreover, comparing two particles with $(en_1/2, 2\pi/e)$ and $(en_2/2, 2\pi/e)$, we have [2]:

$$(e_1 g_1 - e_2 g_2) = 2\pi n \Rightarrow (n_1 - n_2) = 2n \tag{58}$$

Thus n_1, n_2 must be both even or both odd. Thus in the presence of CP symmetry [2]:

$$e_1 = ne \text{ or } e_1 = (n + \frac{1}{2})e \tag{59}$$

To break this constraint, we require breaking CP symmetry. This was part of Witten's motivation [2].

3.1 Abelian Arguments

We shall first give an argument in the Abelian case for clarity and intuition. For this section, let us consider what happens well away from the monopole and only concern ourselves with the usual electromagnetism. Then the Lagrangian density reads:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{\theta e^2}{8\pi^2}\mathbf{E}\cdot\mathbf{B} \tag{60}$$

Consider now the field solutions around a static monopole of charge g, where A^0 is the scalar potential and $\mathbf{A}$ is the vector potential:

$$\mathbf{E} = \boldsymbol{\nabla}A^0, \quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} + g\frac{1}{4\pi r^2}\hat{\mathbf{r}} \tag{61}$$

The contribution of the θ term in the Lagrangian is:

$$\Delta L = \int d^3r \frac{\theta e^2}{8\pi^2}(\boldsymbol{\nabla}A^0(\mathbf{r}))\cdot\left(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}) + g\frac{1}{4\pi r^2}\hat{\mathbf{r}}\right) \tag{62}$$

We can integrate by parts via the identity $(\boldsymbol{\nabla}\psi)\cdot\mathbf{F} = \boldsymbol{\nabla}\cdot\psi\mathbf{F} - \psi\boldsymbol{\nabla}\cdot\mathbf{F}$. This yields:

$$\Delta L = \int d^3r \frac{\theta e^2}{8\pi^2}\boldsymbol{\nabla}\cdot\left(A^0(\mathbf{r})\cdot\left(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}) + g\frac{1}{4\pi r^2}\hat{\mathbf{r}}\right)\right) - \int d^3r \frac{\theta g e^2}{8\pi^2}A^0(\mathbf{r})\boldsymbol{\nabla}\cdot\frac{1}{4\pi r^2}\hat{\mathbf{r}} \tag{63}$$

We see that the term breaks down into a total derivative plus a term dependent on the presence of both a monopole and a nonzero value of θ. If we use Gauss's theorem to convert the volume integral of the total divergence to a surface integral and assuming that it vanishes on an infinite surface far away, we are left with the only contribution from the θ term of the Lagrangian being:

$$\Delta L = -\int d^3r \frac{\theta g e^2}{8\pi^2} A^0(\mathbf{r})\delta^{(3)}(\mathbf{r}) \tag{64}$$

We compare this to the term $A^0(\mathbf{r})J^0(\mathbf{r})$ that would normally appear in the Lagrangian density in the presence of electric charges. We make the identification:

$$J^0(\mathbf{r}) = -\frac{\theta g e^2}{8\pi^2}\delta^{(3)}(\mathbf{r}) \tag{65}$$

which corresponds to an electric point charge of $-\frac{\theta g e^2}{8\pi^2}$ sitting at the origin. We observe that, while the θ term in the absence of a magnetic monopole would have given 0 contribution to the Lagrangian since it is a total derivative, the magnetic monopole acts as an electrically charged object, which of course has a tangible effect on the equations of motion. We also see that, if we use our quantization condition from the previous section that, using the lowest quantization condition $eg = 4\pi$, the electric charge is fractional:

$$q_e = -\frac{\theta e}{2\pi} \tag{66}$$

This is indeed the correct answer, although the argument has some deficiencies since it is ignorant of what happens inside the monopole [4, 6]. One can also examine this value and ask what happens when $\theta = 2\pi$? Then $q_e = -e$ which is non-fractional and could be explained already as a normal electric charge. This supports the idea that θ is a periodic variable.

3.2 Witten Effect in Nonabelian Gauge Theory

Let us now return to the nonabelian gauge theory from before, which does take into account a mechanism for the internal behavior of the monopole. Far from the monopole, let us consider rotations in the U(1) subgroup of SU(2) which is still a good symmetry. A general infinitesimal rotation takes the shape [7]:

$$\phi^a \to (1 + i\alpha^b T^b)\phi^a \tag{67}$$

$$A^a_\mu \to A^a_\mu + \frac{1}{e}\partial_\mu \alpha^a + \varepsilon^{abc} A^b_\mu \alpha^c \tag{68}$$

and if we take $\alpha^a = \epsilon\phi^a/v$, such that the gauge rotation direction is ϕ^a/v and ϵ is infinitesimal, we then have:

$$A^a_\mu \to A^a_\mu + \epsilon\frac{1}{ev}\left(\partial_\mu\phi^a + \varepsilon^{abc}A^b_\mu\phi^c\right) = A^a_\mu + \epsilon\frac{1}{ev}D_\mu\phi^a \tag{69}$$

We consider this transformation under the Noether framework. We use the above to consider an infinitesimal transformation:

$$A^a_\mu \to A^a_\mu + \epsilon \delta A^a_\mu \tag{70}$$

We consider the Noether charge associated with this transformation:

$$Q = \int d^3r \frac{\partial \mathcal{L}}{\partial \partial_0 A^a_\mu} \delta A^a_\mu \tag{71}$$

The derivative is:

$$\frac{\partial}{\partial \partial_0 A^a_\lambda} \mathcal{L} = \frac{\partial}{\partial \partial_0 A^a_\mu} \left(-\frac{1}{4} F^b_{\mu\nu} F^{b\mu\nu} - \frac{\theta e^2}{64\pi^2} \varepsilon^{\mu\nu\sigma\rho} F^b_{\mu\nu} F^b_{\sigma\rho} \right) \tag{72}$$

since the other terms don't depend on the derivatives of A^a_μ. Note that:

$$\frac{\partial}{\partial \partial_0 A^a_\lambda} F^b_{\mu\nu} = \frac{\partial}{\partial \partial_0 A^a_\lambda} \left(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu \right) = \delta_{0\mu} \delta_{\lambda\nu} \delta^{ab} - \delta_{0\nu} \delta_{\mu\lambda} \delta^{ab} \tag{73}$$

since the nonabelian term also does not contribute to the derivative. Then:

$$\begin{aligned} \frac{\partial}{\partial \partial_0 A^a_\lambda} \mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} (\delta^{0\mu} \delta^{\lambda\nu} - \delta^{0\nu} \delta^{\mu\lambda}) - \frac{1}{4} F^{a\mu\nu} (\delta_{0\mu} \delta_{\lambda\nu} - \delta_{0\nu} \delta_{\mu\lambda}) \\ &\quad - \frac{\theta e^2}{64\pi^2} \varepsilon^{\mu\nu\sigma\rho} F^a_{\sigma\rho} (\delta^{0\mu} \delta^{\lambda\nu} - \delta^{0\nu} \delta^{\mu\lambda}) - \frac{\theta e^2}{64\pi^2} \varepsilon^{\mu\nu\sigma\rho} F^a_{\mu\nu} (\delta^{0\sigma} \delta^{\lambda\rho} - \delta^{0\rho} \delta^{\sigma\lambda}) \end{aligned} \tag{74}$$

$$\begin{aligned} &= -\frac{1}{4} F^{a0\lambda} + \frac{1}{4} F^{a\lambda 0} - \frac{1}{4} F^a_{0\lambda} + \frac{1}{4} F^a_{\lambda 0} \\ &\quad - \frac{\theta e^2}{64\pi^2} F^a_{\sigma\rho} (\varepsilon^{0\lambda\sigma\rho} - \varepsilon^{\lambda 0\sigma\rho}) - \frac{\theta e^2}{64\pi^2} F^a_{\mu\nu} (\varepsilon^{\mu\nu 0\lambda} - \varepsilon^{\mu\nu\lambda 0}) \end{aligned} \tag{75}$$

$$= -F^{a0\lambda} - \frac{\theta e^2}{16\pi^2} \varepsilon^{\lambda jk} F^a_{jk} \tag{76}$$

where $\lambda = 1, 2, 3$. Thus, we arrive at the result:

$$\frac{\partial \mathcal{L}}{\partial \partial_0 A^a_\mu} = E^{a\mu} + \frac{\theta e^2}{8\pi^2} B^{a\mu} \tag{77}$$

Then (note the switch to spatial index $i = 1, 2, 3$):

$$Q = \frac{1}{ev} \int d^3r \left(E^{ai} + \frac{\theta e^2}{8\pi^2} B^{ai} \right) D_i \phi^a \tag{78}$$

We identify, based on the fact that in the absence of the θ term the above would give the electric charge, the electric and magnetic charges as:

$$g_e = \frac{1}{v} \int d^3r E^{ai} D_i \phi^a, \quad g_m = \frac{1}{v} \int d^3r B^{ai} D_i \phi^a \tag{79}$$

Thus:

$$Q = \frac{g_e}{e} + \frac{\theta e g_m}{8\pi^2} \tag{80}$$

Since Q is the generator of the symmetry [7], when we do a 2π rotation we must get the identity since we are working in this particular adjoint representation that behaves like a vector [2, 6]:

$$e^{i2\pi Q} = 1 \tag{81}$$

This implies that $Q \equiv n_e \in \mathbb{Z}$. Also invoking the previous quantization condition that $eg_m = 4\pi n_m$, $n_m \in \mathbb{Z}$, we have:

$$g_e = n_e e - \frac{\theta e}{2\pi} n_m \tag{82}$$

$$g_m = \frac{4\pi}{e} n_m \tag{83}$$

Thus we see that there is a fractional electric charge associated with the monopole when there is a nonzero θ term [6].

3.3 Physical Origin via Axion Electrodynamics

One can construct a simple physical picture of why magnetic monopoles have fractional electric charge when θ is nonzero. To do this, we allow θ to have a spacetime dependence, which is relevant to axion electrodynamics [8]. Consider the Abelian case and the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\theta e^2}{64\pi^2} \varepsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} - A_\mu J^\mu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{84}$$

Using the Euler-Lagrange equations [9]:

$$\partial_\beta \frac{\partial}{\partial \partial_\beta A_\alpha} \mathcal{L} = \frac{\partial}{\partial A^\alpha} \mathcal{L} \tag{85}$$

leads to (Appendix C):

$$\partial_\beta F^{\beta\alpha} + \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\beta\alpha} F_{\mu\nu} \partial_\beta \theta = J^\alpha \tag{86}$$

or equivalently (Appendix C):

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho - \frac{e^2}{8\pi^2} \mathbf{B} \cdot \boldsymbol{\nabla} \theta \tag{87a}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \partial_t \mathbf{E} + \mathbf{J} + \frac{e^2}{8\pi^2} \mathbf{B} \partial_t \theta - \frac{e^2}{8\pi^2} \mathbf{E} \times \boldsymbol{\nabla} \theta \tag{87b}$$

Now consider a geometry where a magnetic monopole is surrounded by a region with $\theta = 0$ until some radius R, after which there is a constant value of $\theta = \theta' \neq 0$. Take the integral form of Gauss' Law in the absence of other free charges ($\rho = 0$) which relates the charge of a point or sphere at the origin to the divergence of the electric field:

$$\rho' = \int dV \boldsymbol{\nabla} \cdot \mathbf{E} = -\frac{e^2}{8\pi^2} \int dV \mathbf{B} \cdot \boldsymbol{\nabla}\theta = -\frac{\theta' e^2}{8\pi^2} 4\pi R^2 \mathbf{B}(R) \cdot \hat{\mathbf{r}} = -\frac{\theta' e^2}{8\pi^2}\Phi \tag{88}$$

where Φ is the magnetic flux [8]. We see that the electric charge of the object as seen from inside the $\theta \neq 0$ region is $-\frac{\theta' e^2}{8\pi^2}\Phi$. In fact, if we plug in the form $\mathbf{B} = (g_m/4\pi r^2)\hat{\mathbf{r}}$ for the monopole and impose $eg_m = 4\pi n_m$, then we get:

$$\rho' = -\frac{\theta' e^2 g_m}{8\pi^2} = -\frac{\theta' e}{2\pi} \tag{89}$$

which is the right answer.

An alternative intuition is to consider "Tellegen" constitutive relations (named for B.D.H. Tellegen who proposed them) with the Tellegen parameter $\chi = \theta e^2/8\pi^2$ [10]:

$$\mathbf{D} = (1 + \chi^2)\mathbf{E} + \chi\mathbf{H} \tag{90a}$$
$$\mathbf{B} = \chi\mathbf{E} + \mathbf{H} \tag{90b}$$

or equivalently:

$$\mathbf{D} = \mathbf{E} + \chi\mathbf{B} \tag{91a}$$
$$\mathbf{H} = -\chi\mathbf{E} + \mathbf{B} \tag{91b}$$

Then, Maxwell's Equations become:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho \Rightarrow \boldsymbol{\nabla} \cdot \mathbf{E} = \rho - \mathbf{B} \cdot \boldsymbol{\nabla}\chi \tag{92a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{92b}$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B} \tag{92c}$$
$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J} + \partial_t \mathbf{D} \Rightarrow \boldsymbol{\nabla} \times \mathbf{B} &= \mathbf{J} + \partial_t \mathbf{E} + \mathbf{B}\partial_t\chi + \chi\partial_t\mathbf{B} + \chi\boldsymbol{\nabla} \times \mathbf{E} - \mathbf{E} \times \boldsymbol{\nabla}\chi \\ &\Rightarrow \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \partial_t \mathbf{E} + \mathbf{B}\partial_t\chi - \mathbf{E} \times \boldsymbol{\nabla}\chi\end{aligned} \tag{92d}$$

which are easily seen to be equivalent to Eqs. 87. The physical interpretation of this description is best seen by recalling that:

$$\mathbf{D} = \mathbf{E} + \mathbf{P} \Rightarrow \mathbf{P} = \chi^2\mathbf{E} + \chi\mathbf{H} \tag{93a}$$
$$\mathbf{B} = \mathbf{H} + \mathbf{M} \Rightarrow \mathbf{M} = \chi\mathbf{E} \tag{93b}$$

where $\mathbf{P}$ is the electric polarization and $\mathbf{M}$ is the magnetic polarization. It can now clearly be seen that the effect of θ is to create the so-called "magnetoelectric effect" whereby electric fields induce magnetic dipoles and magnetic fields induce electric dipoles. Thus, when we surround a magnetic monopole with $\theta \neq 0$, the magnetic

fields from the monopole induce electric charges which polarize the rest of the medium, according to:

$$\boldsymbol{\nabla}\cdot\mathbf{P}=\frac{e^2}{8\pi^2}\mathbf{B}\cdot\boldsymbol{\nabla}\theta \tag{94}$$

such that $\mathbf{P}$ is nonzero in the rest of the medium with $\theta\neq 0$. This provides an intuitive understanding of the Witten effect as being related to magnetoelectric polarization of the θ medium, which thus endows magnetic fields with electric charge proportional to θ.

4 Conclusion

We have discussed various aspects of the theory of magnetic monopoles. We began by discussing Dirac's quantization condition, relating the electric charge of an electric monopole and the magnetic charge of a magnetic monopole. We also used electromagnetic duality to relate this quantization condition to one between the electric and magnetic charges of a pair of dyons. We then moved to discussing nonabelian gauge theories with spontaneous symmetry breaking, showing that 't-Hooft-Polyakov magnetic monopoles have a topological origin in such theories. We also demonstrated the Witten effect, stating that when CP symmetry is violated in the Lagrangian via a θ term, magnetic monopoles are endowed with fractional electric charge.

Magnetic monopoles are interesting in various areas of physics. While we discussed an example from cosmology in the introduction, it is also interesting to note that similar constructions relating to dyons can be made in condensed matter physics [11].

A Derivation of Eq. 37

We wish to compute:

$$F^{a\mu\nu}=\partial^\mu A^{a\nu}-\partial_\nu A^{a\mu}+e\varepsilon^{abc}A^{b\mu}A^{c\nu},\quad A^a_\mu=\frac{1}{v}\tilde{A}_\mu\phi^a-\frac{1}{ev^2}\varepsilon^{abc}\phi^b\partial_\mu\phi^c$$

Using that:

$$\partial^\mu(\varepsilon^{abc}\phi^b\partial^\nu\phi^c)-\partial^\nu(\varepsilon^{abc}\phi^b\partial^\mu\phi^c)=\varepsilon^{abc}(\partial^\mu\phi^b)(\partial^\nu\phi^c)-\varepsilon^{abc}(\partial^\nu\phi^b)(\partial^\mu\phi^c)=2\varepsilon^{abc}(\partial^\nu\phi^b)(\partial^\mu\phi^c)$$

and that:

$$\begin{aligned}
\varepsilon^{abc}A^{b\mu}A^{c\nu} &= \varepsilon^{abc}\left(\frac{1}{v}\tilde{A}^{\mu}\phi^{b} - \frac{1}{ev^2}\varepsilon^{bde}\phi^{d}\partial^{\mu}\phi^{e}\right)\left(\frac{1}{v}\tilde{A}^{\nu}\phi^{c} - \frac{1}{ev^2}\varepsilon^{cfg}\phi^{f}\partial^{\nu}\phi^{g}\right)\\
&= \frac{1}{v^2}\tilde{A}^{\mu}\tilde{A}^{\nu}\varepsilon^{abc}\phi^{b}\phi^{c} - \frac{1}{ev^3}\varepsilon^{bca}\varepsilon^{bde}\tilde{A}^{\nu}\phi^{c}\phi^{d}\partial^{\mu}\phi^{e} - \frac{1}{ev^3}\varepsilon^{cab}\varepsilon^{cfg}\tilde{A}^{\mu}\phi^{b}\phi^{f}\partial^{\nu}\phi^{g}\\
&+ \frac{1}{e^2v^4}\varepsilon^{bca}\varepsilon^{bde}\varepsilon^{cfg}(\phi^{d}\partial^{\mu}\phi^{e})(\phi^{f}\partial^{\nu}\phi^{g})\\
&= 0 - \frac{1}{ev^3}(\delta^{cd}\delta^{ae} - \delta^{ce}\delta^{ad})\tilde{A}^{\nu}\phi^{c}\phi^{d}\partial^{\mu}\phi^{e} - \frac{1}{ev^3}(\delta^{af}\delta^{bg} - \delta^{ag}\delta^{bf})\tilde{A}^{\mu}\phi^{b}\phi^{f}\partial^{\nu}\phi^{g}\\
&+ \frac{1}{e^2v^4}(\delta^{cd}\delta^{ae} - \delta^{ce}\delta^{ad})\varepsilon^{cfg}(\phi^{d}\partial^{\mu}\phi^{e})(\phi^{f}\partial^{\nu}\phi^{g})\\
&= -\frac{1}{ev}\tilde{A}^{\nu}\partial^{\mu}\phi^{a} + \frac{1}{ev}\tilde{A}^{\mu}\partial^{\nu}\phi^{a} + \frac{1}{e^2v^4}(\varepsilon^{dfg}\phi^{d}\phi^{f}(\partial^{\mu}\phi^{a})(\partial^{\nu}\phi^{g}) - \varepsilon^{efg}\phi^{a}\phi^{f}(\partial^{\mu}\phi^{e})(\partial^{\nu}\phi^{g}))\\
&= -\frac{1}{ev}\tilde{A}^{\nu}\partial^{\mu}\phi^{a} + \frac{1}{ev}\tilde{A}^{\mu}\partial^{\nu}\phi^{a} - \frac{1}{e^2v^4}\varepsilon^{efg}\phi^{a}\phi^{f}(\partial^{\mu}\phi^{e})(\partial^{\nu}\phi^{g})
\end{aligned}$$

we have:

$$\begin{aligned}
F^{a\mu\nu} &= \frac{1}{v}\partial^{\mu}(\tilde{A}^{\nu}\phi^{a}) - \frac{1}{v}\partial^{\nu}(\tilde{A}^{\mu}\phi^{a}) - \frac{1}{ev^2}2\varepsilon^{abc}(\partial^{\nu}\phi^{b})(\partial^{\mu}\phi^{c})\\
&- \frac{1}{v}\tilde{A}^{\nu}\partial^{\mu}\phi^{a} + \frac{1}{v}\tilde{A}^{\mu}\partial^{\nu}\phi^{a} - \frac{1}{ev^4}\varepsilon^{efg}\phi^{a}\phi^{f}(\partial^{\mu}\phi^{e})(\partial^{\nu}\phi^{g})\\
F^{a\mu\nu} &= \frac{1}{v}\phi^{a}(\partial^{\mu}\tilde{A}^{\nu} - \partial^{\nu}\tilde{A}^{\mu}) - \frac{1}{ev^2}2\varepsilon^{abc}(\partial^{\nu}\phi^{b})(\partial^{\mu}\phi^{c}) - \frac{1}{ev^4}\varepsilon^{efg}\phi^{a}\phi^{f}(\partial^{\mu}\phi^{e})(\partial^{\nu}\phi^{g})
\end{aligned}$$

Note that since ϕ^a is fixed on a sphere, the cross product of its derivatives is parallel to itself, which allows writing $\varepsilon^{abc}(\partial^{\nu}\phi^{b})(\partial^{\mu}\phi^{c}) = \frac{1}{v^2}\phi^{a}(\phi^{d}\varepsilon^{dbc}(\partial^{\nu}\phi^{b})(\partial^{\mu}\phi^{c})) = -\frac{1}{v^2}\phi^{a}(\phi^{f}\varepsilon^{efg}(\partial^{\nu}\phi^{e})(\partial^{\mu}\phi^{g}))$. Then:

$$F^{a\mu\nu} = \frac{1}{v}\phi^{a}\left(\partial^{\mu}\tilde{A}^{\nu} - \partial^{\nu}\tilde{A}^{\mu} - \frac{1}{ev^3}\varepsilon^{efg}\phi^{e}(\partial^{\mu}\phi^{f})(\partial^{\nu}\phi^{g})\right)$$

B Derivation of Eq. 49

We have:

$$M_M = \int r^2 dr d\Omega\left(\frac{1}{4}F^{a}_{jk}F^{a}_{jk} + \frac{1}{2}D_i\phi^{a}D_i\phi^{a} + \frac{\lambda}{4}(\phi^{a}\phi^{a} - v^2)^2\right)$$

with:

$$\phi^{a} = n^{a}\phi(r), \quad A^{a}_{i} = \varepsilon^{abi}n^{b}A(r)$$

Consider first:

$$\begin{aligned}
F^a_{jk} &= \partial_j A^a_k - \partial_k A^a_j + e\varepsilon^{abc} A^b_j A^c_k \\
&= \varepsilon^{abk} n^b \partial_j A(r) + A(r)\varepsilon^{abk}\partial_j n^b - \varepsilon^{abj} n^b \partial_k A(r) - A(r)\varepsilon^{abj}\partial_k n^b + e\varepsilon^{abc}\varepsilon^{bdj} n^d A(r)\varepsilon^{cek} n^e A(r) \\
&= \varepsilon^{abk} n^b n^j \frac{\partial}{\partial r} A(r) + A(r)\varepsilon^{abk}\left(\frac{\delta^{jb}}{r} - \frac{n^j n^b}{r}\right) - \varepsilon^{abj} n^b n^k \frac{\partial}{\partial r} A(r) - A(r)\varepsilon^{abj}\left(\frac{\delta^{kb}}{r} - \frac{n^k n^b}{r}\right) \\
&+ e(\delta^{ae}\delta^{bk} - \delta^{ak}\delta^{be})\varepsilon^{bdj} n^d A(r) n^e A(r) \\
&= (\varepsilon^{abk} n^b n^j - \varepsilon^{abj} n^b n^k)\left(\frac{\partial}{\partial r} A(r) - \frac{1}{r} A(r)\right) + \varepsilon^{ajk}\frac{2}{r} A(r) + e\varepsilon^{kdj} n^d A(r) n^a A(r) \\
&= (\varepsilon^{abk} n^b n^j - \varepsilon^{abj} n^b n^k)\left(\frac{\partial}{\partial r} A(r) - \frac{1}{r} A(r)\right) + \varepsilon^{ajk}\frac{2}{r} A(r) + e\varepsilon^{kdj} n^d n^a A^2(r)
\end{aligned}$$

Let us now evaluate $\frac{1}{4}F^a_{jk}F^a_{jk}$. Since it is rotationally invariant, we shall compute it for $n^a = \delta^{a3}$ which shall be the general result. For the first term to be nonzero, we require $j = 3$ or $k = 3$, which gives nonzero results only when $a = 1, k = 2$ or vice versa. The second term is only nonzero when a, j, k are all different. The third term requires that $a = 3$, and $j = 1, k = 2$ or vice versa. Thus the only nonzero elements are:

$$\begin{aligned}
F^3_{12} &= \frac{2}{r} A(r) + eA^2(r), \quad F^3_{21} = -\frac{2}{r} A(r) - eA^2(r) \\
F^2_{13} &= -\left(\frac{\partial}{\partial r} A(r) - \frac{1}{r} A(r)\right) - \frac{2}{r} A(r), \quad F^2_{31} = \left(\frac{\partial}{\partial r} A(r) - \frac{1}{r} A(r)\right) + \frac{2}{r} A(r) \\
F^1_{23} &= \left(\frac{\partial}{\partial r} A(r) - \frac{1}{r} A(r)\right) + \frac{2}{r} A(r), \quad F^1_{32} = -\left(\frac{\partial}{\partial r} A(r) - \frac{1}{r} A(r)\right) - \frac{2}{r} A(r)
\end{aligned}$$

Thus:

$$\begin{aligned}
\frac{1}{4}F^a_{jk}F^a_{jk} &= \frac{1}{2}\left(\frac{2}{r} A(r) + eA^2(r)\right)^2 + \left(\frac{\partial}{\partial r} A(r) + \frac{1}{r} A(r)\right)^2 \\
&= \frac{3}{r^2} A^2(r) + \frac{2e}{r} A^3(r) + \frac{e^2}{2} A^4(r) + \frac{2}{r} A(r)\frac{\partial}{\partial r} A(r) + \left(\frac{\partial}{\partial r} A(r)\right)^2
\end{aligned}$$

Also:

$$\begin{aligned}
D_i\phi^a = \partial_i\phi^a + e\varepsilon^{abc} A^b_i \phi^c &= \left(\frac{\delta^{ia}}{r} - \frac{n^i n^a}{r}\right)\phi(r) + n^a n^i \frac{\partial}{\partial r}\phi(r) + e\varepsilon^{abc}\varepsilon^{bdi} n^d A(r) n^c \phi(r) \\
&= \left(\frac{\delta^{ia}}{r} - \frac{n^i n^a}{r}\right)\phi(r) + n^a n^i \frac{\partial}{\partial r}\phi(r) - e(\delta^{ad}\delta^{ci} - \delta^{ai}\delta^{cd}) n^d A(r) n^c \phi(r) \\
&= \left(\frac{\delta^{ia}}{r} - \frac{n^i n^a}{r}\right)\phi(r) + n^a n^i \frac{\partial}{\partial r}\phi(r) - e(n^a n^i - \delta^{ai}) A(r)\phi(r) \\
&= \left(\frac{1}{r} + eA(r)\right)(\delta^{ai} - n^a n^i)\phi(r) + n^a n^i \frac{\partial}{\partial r}\phi(r)
\end{aligned}$$

Thus:

$$\begin{aligned}\frac{1}{2}D_i\phi^a D_i\phi^a &= \frac{1}{2}\bigg(\bigg(\frac{1}{r}+eA(r)\bigg)(\delta^{ai}-n^a n^i)\phi(r)+n^a n^i\frac{\partial}{\partial r}\phi(r)\bigg)\\ &\times\bigg(\bigg(\frac{1}{r}+eA(r)\bigg)(\delta^{ai}-n^a n^i)\phi(r)+n^a n^i\frac{\partial}{\partial r}\phi(r)\bigg)\\ &= \bigg(\frac{1}{r}+eA(r)\bigg)^2\phi(r)^2+\frac{1}{2}\bigg(\frac{\partial}{\partial r}\phi(r)\bigg)^2\end{aligned}$$

Also:

$$\frac{\lambda}{4}(n^a\phi(r)n^a\phi(r)-v^2)^2=\frac{\lambda}{4}\phi(r)^4-\frac{\lambda}{2}\phi(r)^2v^2+\frac{\lambda}{4}v^4$$

Putting it all together:

$$\begin{aligned}M_M = 4\pi\int_0^\infty r^2dr(&\frac{3}{r^2}A^2(r)+\frac{2e}{r}A^3(r)+\frac{e^2}{2}A^4(r)+\frac{2}{r}A(r)\frac{\partial}{\partial r}A(r)+\bigg(\frac{\partial}{\partial r}A(r)\bigg)^2\\ &+\bigg(\frac{1}{r}+eA(r)\bigg)^2\phi(r)^2+\frac{1}{2}\bigg(\frac{\partial}{\partial r}\phi(r)\bigg)^2+\frac{\lambda}{4}\phi(r)^4-\frac{\lambda}{2}\phi(r)^2v^2+\frac{\lambda}{4}v^4)\end{aligned}$$

Let us make the substitutions:

$$A(r)=v\rho a(\rho),\quad \phi(r)=v\rho f(\rho),\quad \rho=evr,\quad \beta=\frac{\lambda}{e^2}$$

Giving:

$$\begin{aligned}M_M = 4\pi\int_0^\infty &\frac{1}{e^3v^3}\rho^2d\rho(3e^2v^4a^2(\rho)+2e^2v^4\rho^2a^3(\rho)\\ &+\frac{e^2v^4\rho^4}{2}a^4(\rho)+2e^2v^4a(\rho)\frac{\partial}{\partial\rho}(\rho a(\rho))+e^2v^4\bigg(\frac{\partial}{\partial\rho}\rho a(\rho)\bigg)^2\\ &+e^2v^4\bigg(\frac{1}{\rho}+\rho a(\rho)\bigg)^2\rho^2f(\rho)^2+\frac{e^2v^4}{2}\bigg(\frac{\partial}{\partial\rho}\rho f(\rho)\bigg)^2+\frac{\lambda}{4}v^4\rho^4f(\rho)^4-\frac{\lambda}{2}v^4\rho^2f(\rho)^2+\frac{\lambda}{4}v^4)\\ M_M = \frac{4\pi M_W}{e^2}\int_0^\infty &\rho^2d\rho(6a^2(\rho)+2\rho^2a^3(\rho)+\frac{\rho^4}{2}a^4(\rho)+4\rho a(\rho)\frac{\partial}{\partial\rho}a(\rho)+\rho^2\bigg(\frac{\partial}{\partial\rho}a(\rho)\bigg)^2\\ &+\frac{3}{2}f(\rho)^2+2\rho^2a(\rho)f(\rho)^2+\rho^4a(\rho)^2f(\rho)^2+\rho f(\rho)\frac{\partial}{\partial\rho}f(\rho)+\frac{1}{2}\rho^2\bigg(\frac{\partial}{\partial\rho}f(\rho)\bigg)^2\\ &+\frac{\beta}{4}\rho^4f(\rho)^4-\frac{\beta}{2}\rho^2f(\rho)^2+\frac{\beta}{4})\end{aligned}$$

This is the same expression as in [1]. We use trial functions to minimize the integral for a value β. The trial functions are chosen such that they obey $\phi(r)\to v$ and $A(r)\to -1/er$ as $r\to\infty$, implying $f(r)\to 1/\rho$ and $a(\rho)\to -1/\rho^2$. We perform this minimization numerically.

C Derivation of Eq. 87

We have:

$$\begin{aligned}&\partial_\beta \frac{\partial}{\partial \partial_\beta A_\alpha}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} - A_\mu J^\mu\right)\\ &= \frac{\partial}{\partial A^\alpha}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} - A_\mu J^\mu\right)\end{aligned}$$

The right hand side is:

$$\frac{\partial}{\partial A^\alpha}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} - A_\mu J^\mu\right) = -J^\alpha$$

The first term on the left hand side is:

$$\begin{aligned}\partial_\beta \frac{\partial}{\partial \partial_\beta A_\alpha}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) &= -\frac{1}{4}\partial_\beta\left(F^{\mu\nu}\frac{\partial}{\partial\partial_\beta A_\alpha}(\partial_\mu A_\nu - \partial_\nu A_\mu) + F_{\mu\nu}\frac{\partial}{\partial\partial_\beta A_\alpha}(\partial^\mu A^\nu - \partial^\nu A^\mu)\right)\\ &= -\frac{1}{4}\partial_\beta\big(F^{\mu\nu}(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu) + F_{\mu\nu}(\delta^{\beta\mu}\delta^{\alpha\nu} - \delta^{\beta\nu}\delta^{\alpha\mu})\big)\\ &= -\frac{1}{2}\partial_\beta\big(F^{\beta\alpha} - F^{\alpha\beta}\big) = -\partial_\beta F^{\beta\alpha}\end{aligned}$$

where in the last step we have used that $F^{\alpha\beta}$ is antisymmetric. The second term on the left hand side is:

$$\begin{aligned}&\partial_\beta \frac{\partial}{\partial \partial_\beta A_\alpha}\left(\frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}\right)\\ &= \frac{e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}\partial_\beta\left(\theta F_{\sigma\rho}\frac{\partial}{\partial\partial_\beta A_\alpha}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \theta F_{\mu\nu}\frac{\partial}{\partial\partial_\beta A_\alpha}(\partial_\sigma A_\rho - \partial_\rho A_\sigma)\right)\\ &= \frac{e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}\partial_\beta\big(\theta F_{\sigma\rho}(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu) + \theta F_{\mu\nu}(\delta^\beta_\sigma\delta^\alpha_\rho - \delta^\beta_\rho\delta^\alpha_\sigma)\big)\\ &= \frac{e^2}{64\pi^2}\partial_\beta\big(\theta(\varepsilon^{\beta\alpha\sigma\rho} - \varepsilon^{\alpha\beta\sigma\rho})F_{\sigma\rho} + \theta(\varepsilon^{\mu\nu\beta\alpha} - \varepsilon^{\mu\nu\alpha\beta})F_{\mu\nu}\big)\\ &= \frac{\theta e^2}{16\pi^2}\varepsilon^{\mu\nu\beta\alpha}\partial_\beta F_{\mu\nu} + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\beta\alpha}F_{\mu\nu}\partial_\beta\theta\\ &= \frac{\theta e^2}{16\pi^2}(\varepsilon^{\mu\nu\beta\alpha}\partial_\beta\partial_\mu A_\nu - \varepsilon^{\mu\nu\beta\alpha}\partial_\beta\partial_\nu A_\mu) + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\beta\alpha}F_{\mu\nu}\partial_\beta\theta\\ &= \frac{\theta e^2}{16\pi^2}(-2\varepsilon^{\mu\nu\beta\alpha}\partial_\beta\partial_\nu A_\mu) + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\beta\alpha}F_{\mu\nu}\partial_\beta\theta\\ &= \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\beta\alpha}F_{\mu\nu}\partial_\beta\theta\end{aligned}$$

where in the last step we have used that the first term is equal to its own negative. The third term is zero. Thus we have:

$$\partial_\beta F^{\beta\alpha} + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\beta\alpha}F_{\mu\nu}\partial_\beta\theta = J^\alpha$$

This yields Maxwell's equations:

$$\begin{aligned}
\partial_i F^{i0} + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu i0}F_{\mu\nu}\partial_i\theta &= J^0 \\
\Rightarrow \partial_i E^i + \frac{e^2}{16\pi^2}\varepsilon^{jki0}F_{jk}\partial_i\theta &= \rho \\
\Rightarrow \boldsymbol{\nabla}\cdot\mathbf{E} + \frac{e^2}{8\pi^2}\mathbf{B}\cdot\boldsymbol{\nabla}\theta &= \rho \\
\partial_0 F^{0i} + \partial_j F^{ji} + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu 0i}F_{\mu\nu}\partial_0\theta + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu ji}F_{\mu\nu}\partial_j\theta &= J^i \\
\Rightarrow -\partial_0 E^i + \varepsilon^{ijk}\partial_j B^k - \frac{e^2}{8\pi^2}B^i\partial_0\theta - \frac{e^2}{8\pi^2}\varepsilon^{ijk}E^k\partial_j\theta &= J^i \\
\Rightarrow \boldsymbol{\nabla}\times\mathbf{B} = \partial_t\mathbf{E} + \mathbf{J} + \frac{e^2}{8\pi^2}\mathbf{B}\partial_t\theta - \frac{e^2}{8\pi^2}\mathbf{E}\times\boldsymbol{\nabla}\theta &
\end{aligned}$$

where we have used that $\varepsilon^{imn}B^i = -\frac{1}{2}\varepsilon^{imn}\varepsilon^{ijk}F_{jk} = -F_{mn}$.

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AI Statement ChatGPT was used conversationally to develop a stronger understanding of some concepts and to check or clarify some derivations. Its use was to help understand a particular thing or to help check or clarify a particular part of the derivation; it was not used to generate any part of the text or equations, and I composed and wrote the paper myself. AI also was used in assisting the numerical aspects of this project: the numerics in Fig. 4, and generation of field configurations for Figs. 2,3.

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