‘t Hooft-Polyakov Monopoles and the Witten Effect

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Motivating Magnetic Monopoles

- It would seem magnetic monopoles don't exist: $\nabla \cdot \mathbf{B} = 0$
- But gauge ambiguity => Dirac's Quantization Condition

- Interior of monopoles? 't Hooft-Polyakov monopoles in many nonabelian gauge theories: spontaneous symmetry breaking + topological winding

- Witten Effect: magnetic monopoles possess fractional electric charge in the presence of a CP-breaking term.
- Motivated by EM duality, dyons: electric and magnetic charge, analogous quantization.

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

$$\begin{pmatrix} \rho_e' \\ \rho_m' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}$$

Dirac's Quantization Condition

- We want to create the field: $\mathbf{B}(\mathbf{r}) = \frac{g}{4\pi r^2}\hat{\mathbf{r}}$
- Use piecewise **A(r)**, which are related in region of overlap via gauge transformation:

$$\mathbf{A}_N(\mathbf{r}) = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\varphi}, \quad \mathbf{A}_S(\mathbf{r}) = -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\hat{\varphi}$$

$$\mathbf{A}_N - \mathbf{A}_S = -\boldsymbol{\nabla}\chi(\mathbf{r})$$

$$\chi(\mathbf{r}) = -\frac{g}{2\pi}\varphi$$

- Require that $e^{-ie\chi(\varphi=0)} = e^{-ie\chi(\varphi=2\pi)}$ (single-valued) since:

$$\psi(\mathbf{r}) \to e^{-ie\chi(\mathbf{r})}\psi(\mathbf{r}), \quad \mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla}\chi(\mathbf{r}) = \mathbf{A} - \frac{i}{e}e^{ie\chi(\mathbf{r})}\boldsymbol{\nabla}e^{-ie\chi(\mathbf{r})}$$

$$\Rightarrow \frac{eg}{2\pi} = n \in \mathbb{Z}$$

‘t Hooft’s Motivation: Breaking Flux Tubes

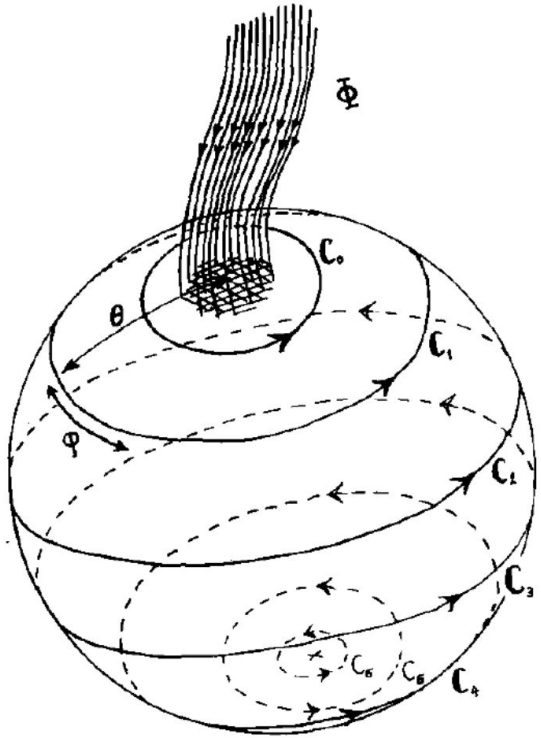
- If flux tube can be broken (by a monopole), **A**(**r**) can be smoothly deformed to a trivial winding far away from the end of the tube.
- This CANNOT be done remaining in U(1).

$$S^1 \to S^1 \qquad \pi_1(S^1) = \mathbb{Z}$$

- But it CAN be done if near the monopole we can take the path through a larger gauge group, eg. SU(2)!

$$S^1 \to S^3 \qquad \pi_1(S^3) = 0$$

- So consider monopoles in nonabelian gauge theory.



[1]

[1] G. ’t Hooft, “Magnetic monopoles in unified gauge theories,” Nuclear Physics B, vol. 79, pp. 276–284, 1974.

Nonabelian Gauge Theory with SSB

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\phi^a D_\mu\phi^a - \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2$$

where $\quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon^{abc}A^b_\mu A^c_\nu, \quad D_\mu\phi^a = \partial_\mu\phi^a + e\varepsilon^{abc}A^b_\mu\phi^c$

where $a, b, c = 1, 2, 3$ label the adjoint representation of SU(2)

Vacuum condition: $\langle\phi^a\phi^a\rangle = v^2$

Then:

- SU(2) symmetry broken down to U(1)
- Two components of gauge field become massive

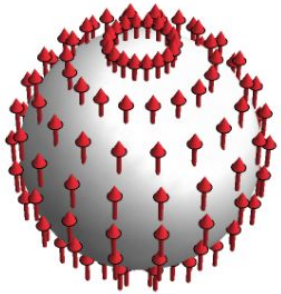
$$\mathcal{L} \supset \frac{1}{2}v^2e^2(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}) \;\Rightarrow\; M_W = ev$$

Energy Density and Topology

$$\Theta_{00} = \frac{1}{2}(\mathbf{E}^a \cdot \mathbf{E}^a + \mathbf{B}^a \cdot \mathbf{B}^a + D_0\phi^a D_0\phi^a + D_i\phi^a D_i\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2,$$

$$E^{ai} = -F^{a0i}, \quad B^{ai} = -\frac{1}{2}\varepsilon^{ijk}F^a_{jk}$$

- Finite energy excitations must look like vacuum far from the origin.
- Vacuum: $\langle\phi^a\phi^a\rangle = v^2$ is two-sphere.
- Mapping from spatial two-sphere to vacuum two-sphere: $\pi_2(S^2) = \mathbb{Z}$



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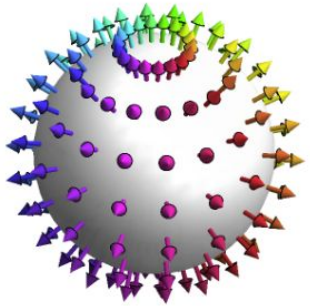

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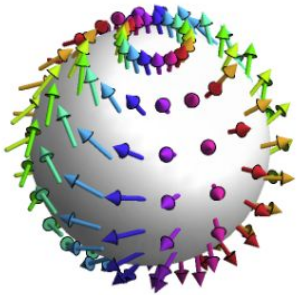

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Gauge Fields to Minimize the Energy Density

Energy density would diverge if A=0. $\int d^3x\Theta_{00} \geq \int d^3x\frac{1}{2}(\hat{\mathbf{r}}\times\boldsymbol{\nabla}\phi^a)^2 \geq \int r^2 dr d\Omega\frac{1}{r^2}$

To minimize the term with the covariant derivatives, A should cancel spatial derivatives:

$$D_\mu n^a = \partial_\mu n^a + e\varepsilon^{abc}A^b_\mu n^c = 0 \qquad\qquad \phi^a = vn^a$$

Decompose into parallel and perpendicular parts:

$$A^a_\mu = (A^b_i n^b)n^a + \mathcal{A}^a_i$$

The perpendicular part is fixed but the parallel part is free:

$$A^a_\mu = \frac{1}{v}\tilde{A}_\mu\phi^a - \frac{1}{ev^2}\varepsilon^{abc}\phi^b\partial_\mu\phi^c$$

Corresponding Magnetic Field and Charge

Then:

$$F^{a\mu\nu} = \frac{1}{v}\phi^a\left(\partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu - \frac{1}{ev^3}\varepsilon^{efg}\phi^e(\partial^\mu\phi^f)(\partial^\nu\phi^g)\right)$$

And magnetic field:

$$B^i = -\frac{1}{2}\varepsilon^{ijk}\left(\partial^j \tilde{A}^k - \partial^k \tilde{A}^j - \frac{1}{ev^3}\varepsilon^{efg}\phi^e(\partial^j\phi^f)(\partial^k\phi^g)\right)$$

$$E^{ai} = -F^{a0i}, \quad B^{ai} = -\frac{1}{2}\varepsilon^{ijk}F^a_{jk}$$

Magnetic charge:

$$g = \int_S B^i dS^i$$

Dirac quantization condition! (factor of 2 is because electric charge in units of e/2 here due to U(1) generator) [1]

Use winding number of map between two two-spheres:

$$N = \frac{1}{8\pi}\int_S dS_i \varepsilon^{ijk}\varepsilon^{efg}\tilde{\phi}^e(\partial^j\tilde{\phi}^f)(\partial^k\tilde{\phi}^g)$$

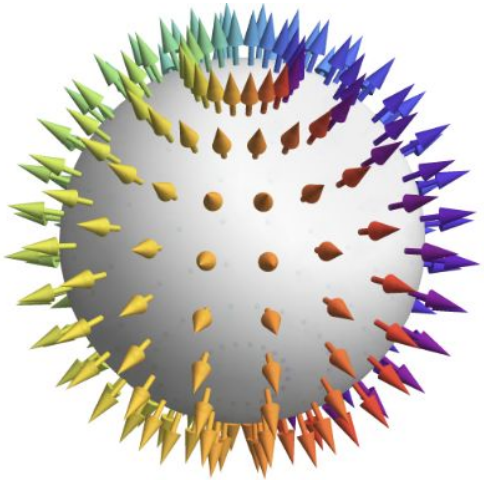
$$\tilde{\phi}^a = \phi^a/v$$

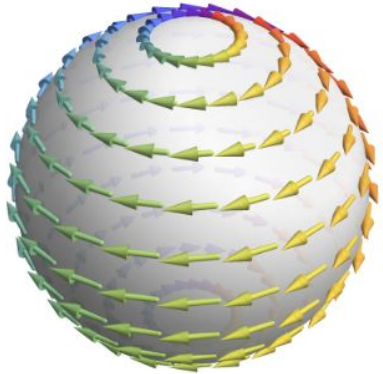
$$\Rightarrow \quad g = \frac{4\pi N}{e}$$

[1] J. A. Harvey, "Magnetic monopoles, duality and symmetry," Lecture Notes, arXiv:hep-th/9603086, 1996

Hedgehog Configuration

$$\phi^a(t,\mathbf{r}) = v\delta^{ai}n^i = vn^a$$





$$A_i^a = -\frac{1}{er}\varepsilon^{abi}n^b$$

$$F^{ajk} = -\frac{1}{er^2}n^a\varepsilon^{ejk}n^e$$

$$B^i = \frac{1}{er^2}n^i$$

$$g = \int_S B^i dS^i = (4\pi r^2)\frac{1}{er^2} = \frac{4\pi}{e}$$

Mass of Monopole

- Expect the massive gauge fields to decay as $e^{-M_W r}$
- Can lower bound the mass using fields outside of this [1]:

$$M_M \geq 4\pi \int_{1/M_W}^{\infty} r^2 dr \frac{1}{2}(B^i B^i) = \frac{2\pi}{e^2} \int_{1/M_W}^{\infty} dr \frac{1}{r^2} = \frac{2\pi M_W}{e^2} = \frac{2\pi v}{e}$$

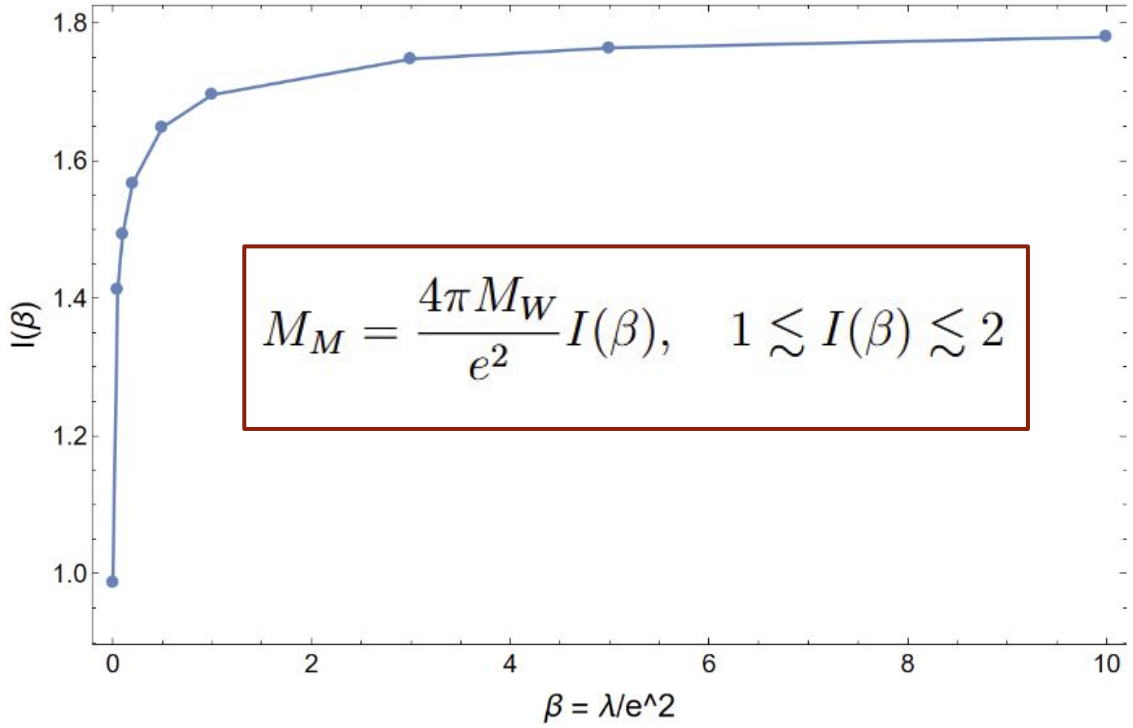
- More generally:

$$\phi(\mathbf{r}) = n^a v \rho f(\rho), \quad A_i^a = \varepsilon^{abi} n^b v \rho a(\rho), \quad \rho = ver, \quad \beta = \frac{\lambda}{e^2}$$

$$\begin{aligned} M_M = -L = \frac{4\pi M_W}{e^2} \int_0^{\infty} & \rho^2 d\rho (6a^2(\rho) + 2\rho^2 a^3(\rho) + \frac{\rho^4}{2} a^4(\rho) + 4\rho a(\rho) \frac{\partial}{\partial \rho} a(\rho) \\ & + \rho^2 \left(\frac{\partial}{\partial \rho} a(\rho) \right)^2 + \frac{3}{2} f(\rho)^2 + 2\rho^2 a(\rho) f(\rho)^2 + \rho^4 a(\rho)^2 f(\rho)^2 \\ & + \rho f(\rho) \frac{\partial}{\partial \rho} f(\rho) + \frac{1}{2} \rho^2 \left(\frac{\partial}{\partial \rho} f(\rho) \right)^2 + \frac{\beta}{4} \rho^4 f(\rho)^4 - \frac{\beta}{2} \rho^2 f(\rho)^2 + \frac{\beta}{4}) \end{aligned}$$

[1] S. Coleman, "The magnetic monopole fifty years later," in Les Houches Summer School of Theoretical Physics: Laser-Plasma Interactions, pp. 461–552, 1982.

Numerically Minimize the Integral



$$M_M = \frac{4\pi M_W}{e^2} I(\beta), \quad 1 \lesssim I(\beta) \lesssim 2$$

Thus monopoles are much heavier than the gauge bosons of the theory!

Monopoles hard to create in accelerators / present universe, but perhaps in early universe?

General Topological Arguments

- Gauge group G undergoes SSB to gauge group H.
- Let ϕ_0 be one of the degenerate minima of $U(\phi)$
- It is transformed to the other minima by G/H.
- Thus far away from the monopole, the mapping is from spatial two-sphere to G/H.
- Characterized by $\pi_2(G/H)$

- How to connect this surface topology in ‘t Hooft’s arguments to loop topology in Dirac’s arguments?

General Topological Arguments

Consider the gauge transformations: $\Omega_N(\theta,\varphi)\phi(\theta,\varphi) = \phi_0, \quad 0 \le \theta \le \pi/2$

$\Omega_S(\theta,\varphi)\phi(\theta,\varphi) = \phi_0, \quad \pi/2 \le \theta \le \pi$

On the equator: $\Omega_N(\pi/2,\varphi)\Omega_S^{-1}(\pi/2,\varphi)\phi_0 = \phi_0 \quad \Rightarrow \quad \Omega_N(\pi/2,\varphi)\Omega_S^{-1}(\pi/2,\varphi) \in H$

Thus we associated a mapping from S^2 to G/H with a mapping from S^1 to H

In fact, $\pi_2(G/H)$ is isomorphic to $\pi_1(H)$ [1]

Thus the ‘t Hooft and Dirac arguments are related!

Previous e.g.: $\pi_2(S^2) = \pi_1(S^1) = \mathbb{Z}$

No singularity! Loops cannot be contracted within H far from the origin but can be contracted within G near the monopole. Kernel of:

$$\pi_1(H) \to \pi_1(G)$$

[1] J. Preskill, “Magnetic monopoles,” Ann. Rev. Nucl. Part. Sci., vol. 34, pp. 461–530, 1984.

Witten Effect

Consider adding a term to the Lagrangian: $\Delta\mathcal{L} = -\frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F^a_{\mu\nu}F^a_{\sigma\rho}$

Can also obtain this via duality transformation (consider Abelian case):

$$\begin{aligned}(\mathbf{E}^2 - \mathbf{B}^2) \to ((\mathbf{E}\cos\alpha + \mathbf{B}\sin\alpha)^2 - (\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha)^2) \\ = \cos 2\alpha(\mathbf{E}^2 - \mathbf{B}^2) - 2\sin 2\alpha(\mathbf{E}\cdot\mathbf{B})\end{aligned}$$

Thus we expect the magnetic monopole to acquire electric charge as well.

Abelian Case

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{\theta e^2}{8\pi^2}\mathbf{E}\cdot\mathbf{B}$$

Consider in the presence of a monopole: $\mathbf{E} = \boldsymbol{\nabla} A^0, \quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} + g\frac{1}{4\pi r^2}\hat{\mathbf{r}}$

$$\Delta L = \int d^3r \frac{\theta e^2}{8\pi^2}(\boldsymbol{\nabla}A^0(\mathbf{r}))\cdot\left(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}) + g\frac{1}{4\pi r^2}\hat{\mathbf{r}}\right) = -\int d^3r \frac{\theta g e^2}{8\pi^2}A^0(\mathbf{r})\delta^{(3)}(\mathbf{r})$$

Thus charge density:

$$J^0(\mathbf{r}) = -\frac{\theta g e^2}{8\pi^2}\delta^{(3)}(\mathbf{r})$$

$$\Rightarrow \quad q_e = -\frac{\theta e}{2\pi}$$

Nonabelian Case

General infinitesimal rotation:

$$\phi^a \to (1 + i\alpha^b T^b)\phi^a$$

$$A^a_\mu \to A^a_\mu + \underbrace{\frac{1}{e}\partial_\mu \alpha^a + \varepsilon^{abc} A^b_\mu \alpha^c}_{\epsilon \delta A^a_\mu}$$

Consider Noether charge:

$$\frac{\partial}{\partial \partial_0 A^a_\lambda}\mathcal{L} = \frac{\partial}{\partial \partial_0 A^a_\mu}\left(-\frac{1}{4}F^b_{\mu\nu}F^{b\mu\nu} - \frac{\theta e^2}{64\pi^2}\varepsilon^{\mu\nu\sigma\rho}F^b_{\mu\nu}F^b_{\sigma\rho}\right) = E^{a\mu} + \frac{\theta e^2}{8\pi^2}B^{a\mu}$$

$$Q = \int d^3r \frac{\partial \mathcal{L}}{\partial \partial_0 A^a_\mu}\delta A^a_\mu = \frac{1}{ev}\int d^3r \left(E^{ai} + \frac{\theta e^2}{8\pi^2}B^{ai}\right) D_i \phi^a$$

We have:

$$g_e = \frac{1}{v}\int d^3r E^{ai} D_i \phi^a, \quad g_m = \frac{1}{v}\int d^3r B^{ai} D_i \phi^a$$

$$Q = \frac{g_e}{e} + \frac{\theta e g_m}{8\pi^2}$$

since Q generates symmetry, $e^{i2\pi Q} = 1 \qquad Q \equiv n_e \in \mathbb{Z}$

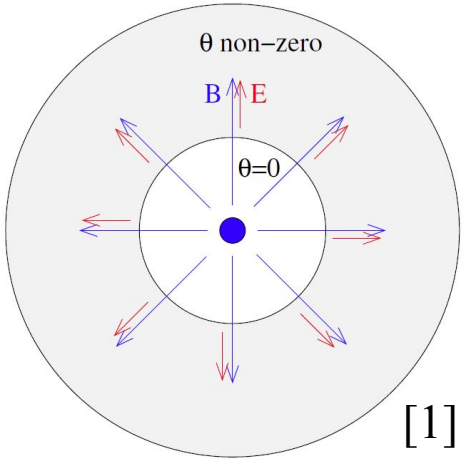
$$\Rightarrow \begin{aligned} g_e &= n_e e - \frac{\theta e}{2\pi} n_m \\ g_m &= \frac{4\pi}{e} n_m \end{aligned}$$

Physical Origin via Axion Electrodynamics

Maxwell's Equations (Abelian) become:

$$\nabla \cdot \mathbf{E} = \rho - \frac{e^2}{8\pi^2}\mathbf{B} \cdot \nabla\theta$$

$$\nabla \times \mathbf{B} = \partial_t \mathbf{E} + \mathbf{J} + \frac{e^2}{8\pi^2}\mathbf{B}\partial_t\theta - \frac{e^2}{8\pi^2}\mathbf{E} \times \nabla\theta$$



[1]

$$\rho' = \int dV \nabla \cdot \mathbf{E} = -\frac{e^2}{8\pi^2}\int dV \mathbf{B} \cdot \nabla\theta = -\frac{\theta' e^2}{8\pi^2}\Phi$$

Alternatively, the "Tellegen" constitutive relations give the same Maxwell's equations.:
Magnetic fields creating electric polarization and vice versa!

$$\mathbf{D} = \mathbf{E} + \mathbf{P} \Rightarrow \mathbf{P} = \chi^2\mathbf{E} + \chi\mathbf{H}$$

$$\mathbf{B} = \mathbf{H} + \mathbf{M} \Rightarrow \mathbf{M} = \chi\mathbf{E}$$

[1] D. Tong, "Gauge theory," Lecture Notes, davidtong.org/teaching/gauge-theory, 2018.

Summary

- ‘t Hooft-Polyakov: there exist stable (topological) and finite energy (gauge field cancels spatial derivatives) excitations of nonabelian gauge theories; the winding number is the monopole charge.
- Witten: adding a CP-breaking term to the Lagrangian endows magnetic monopoles with fractional electric charge.

References

[1] G. ’t Hooft, “Magnetic monopoles in unified gauge theories,” Nuclear Physics B, vol. 79, pp. 276–284, 1974.

[2] E. Witten, “Dyons of charge eθ/2π,” Physics Letters, vol. 86B, pp. 283–287, 1979.

[3] D. Tong, “Gauge theory,” Lecture Notes, davidtong.org/teaching/gauge-theory, 2018.

[4] S. Coleman, “The magnetic monopole fifty years later,” in Les Houches Summer School of Theoretical Physics: Laser-Plasma Interactions, pp. 461–552, 1982.

[5] J. Preskill, “Magnetic monopoles,” Ann. Rev. Nucl. Part. Sci., vol. 34, pp. 461–530, 1984.

[6] J. A. Harvey, “Magnetic monopoles, duality and symmetry,” Lecture Notes, arXiv:hep-th/9603086, 1996.

[7] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory. CRC Press, 2018.

[8] F. Wilczek, “Two applications of axion electrodynamics,” Physical Review Letters, vol. 58, no. 18, pp. 1799–1802, 1987.

[9] J. D. Jackson, Classical Electrodynamics. John Wiley and Sons, 3 ed., 1999.

[10] B. D. H. Tellegen, “The gyrator, a new electric network element,” Philips Research Reports, vol. 3, pp. 81–101, 1948.

[11] X.-L. Qi, R. Li, J. Zang, and S.-C. Zhang, “Inducing a magnetic monopole with topological surface states,” Science, vol. 323, no. 5918, pp. 1184–1187, 2009.

Supplement: Duality and Dyons

Maxwell's Equations:

$$\begin{pmatrix} \boldsymbol{\nabla}\cdot & 0 \\ 0 & \boldsymbol{\nabla}\cdot \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\nabla}\times & 0 \\ 0 & \boldsymbol{\nabla}\times \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \partial_t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}$$

Duality transformations:

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$ and same for charges and currents

If electric and magnetic monopoles exist, so do dyons.

Dirac's quantization condition gets modified as:

Duality rotated electric and magnetic monopoles:

$$(e_1, g_1) \equiv (e\cos\alpha, -e\sin\alpha)$$
$$(e_2, g_2) \equiv (g\sin\alpha, g\cos\alpha)$$

$$\Rightarrow \quad e_1 g_2 - e_2 g_1 = eg = 2\pi n$$

Witten effect!

$$e_i = n_i e + m_i f$$

eg. solution:

$$g_i = \frac{2\pi}{e} m_i$$

Supplement: CP symmetry

If there is CP symmetry both the particle and its CP transformed version are valid and obey a quantization relation: $(e_1 g_1 - (-e_1) g_1) = 2\pi n \Rightarrow 2e_1 = en$

Also compare $(en_1/2, 2\pi/e)$ and $(en_2/2, 2\pi/e)$

$$(e_1 g_1 - e_2 g_2) = 2\pi n \Rightarrow (n_1 - n_2) = 2n$$

Thus CP symmetry forces:

$$e_1 = ne \text{ or } e_1 = (n + \frac{1}{2})e$$

No fractional electric charge.